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MATHEMATICS *today*

INDIA'S NUMBER 1 MATHEMATICS MONTHLY

ENTRANCE EXAMINATIONS SPECIAL
FOR IIT-JEE & ENGINEERING ENTRANCE EXAMS

IIT-JEE '97 PRACTICE TEST SERIES-IV

PROBLEMS OF THE MONTH

VECTORS

RUSSIAN CONTEST PROBLEMS

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IIT-JEE '97

PRACTICE TEST SERIES-IV

Time : Three hours

Maximum marks : 100

Note: (1) There are 12 questions in this paper

(2) Attempt all questions

(3) There is no negative marking

(4) Use of a calculator, slide rule, graph paper and logarithmic, trigonometric and statistical table is not permitted.

PART A

1. There are ten parts in this question. Each part has one or more than one correct answer(s). For each part, write the letters from (a), (b), (c) (d) corresponding to the correct answer. [10 × 2]

(i) If m and n are positive integers, then smaller of $\sqrt[m]{m}$ and $\sqrt[n]{n}$ cannot exceed-

- (a) 0.5 (b) 0.75
(c) 1.0 (d) none of these.

(ii) In a triangle with one angle of $\frac{2\pi}{3}$, the lengths of the sides form an A.P. If the length of the greatest side is 7 cm., the area of the triangle is

- (a) $3\sqrt{15}/4$ (b) $15\sqrt{3}/4$
(c) $15/4 \text{ cm}^2$ (d) $3\sqrt{3}/4$

(iii). A root of the equation, $\sin x - x + 1 = 0$, lies in

- (a) $[0, \pi/2]$ (b) $[\pi/2, \pi]$
(c) $[\pi, 3\pi/2]$ (d) $[3\pi/2, 2\pi]$

(iv) A sum of money is round off to the nearest rupee. The probability that the round of error is at most ten paise is

- (a) 19/101 (b) 10/101
(c) 9/101 (d) none of these.

(v) The number of pairs (x, y) satisfying the equations, $\sin x + \sin y = \sin(x + y)$ and

$|x| + |y| = 1$, is

- (a) 2 (b) 4
(c) 6 (d) infinite.

(vi) If $f(x) = \begin{cases} x[x + \frac{1}{2} \sin(\ln x^2)], & x \neq 0 \\ 0 & x = 0; \end{cases}$

then f is

- (a) continuous at $x = 0$
(b) continuous everywhere
(c) both continuous and derivable at $x = 0$
(d) continuous but not derivable at $x = 0$.

(vii) If $|z| z + az + 1 = 0$ ($a > 0$), then z is

- (a) purely real (b) purely imaginary
(c) a negative real number
(d) none of these.

(viii) Let the unit vectors $\vec{a}$ and $\vec{b}$ be perpendicular and the unit vector $\vec{c}$ be inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If

$$\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a} \times \vec{b}); \text{ then}$$

- (a) $\alpha = \beta = \cos\theta, \gamma^2 = -\cos 2\theta$
(b) $\alpha = \beta = \cos\theta, \gamma^2 = \cos 2\theta$
(c) $\alpha = \cos\theta, \beta = \sin\theta, \gamma^2 = \cos 2\theta$
(d) $\alpha = \sin\theta, \beta = \cos\theta, \gamma^2 = -\cos 2\theta$.

(ix) The tangents to the curve,

$$x = a(\theta - \sin\theta), y = a(1 + \cos\theta);$$

at the points, $\theta = (2k + 1)\pi$, ($k \in \mathbb{I}$) are all parallel to

- (a) the line, $y = x$ (b) the line, $y = -x$
 (c) the x-axis (d) the y-axis.

(x) The volume of the parallelopiped whose sides are given by

$$\vec{OA} = (\lambda + 2)\hat{i} + (\lambda + 1)(\lambda + 2)\hat{j} + \hat{k}$$

$$\vec{OB} = (\lambda + 3)\hat{i} + (\lambda + 2)(\lambda + 3)\hat{j} + \hat{k} \text{ and}$$

$$\vec{OC} = (\lambda + 4)\hat{i} + (\lambda + 3)(\lambda + 4)\hat{j} + \hat{k} \text{ is}$$

- (a) 2 (b) 4λ
 (c) $\lambda + 3$ (d) none of these.

2. This question contains ten incomplete statements. Fill in the blanks so that the statements are correct. Write only the answers.

(i) The set of all x in the interval $[0, \pi/2]$ for which $\log_{(1/2)} \cos x < \log_{(1/2)} \tan x$ is....

(ii) If A_i is the centre of the circle, $x^2 + y^2 + 2g_i x + 5 = 0$ and t_i is the length of the tangent from any point to this circle, $i = 1, 2, 3$; then $A_1 A_2 \cdot t_3^2 + A_2 A_3 \cdot t_1^2 + A_3 A_1 \cdot t_2^2 = \dots$

(iii) If an equilateral triangle and a regular hexagon have the same perimeter, then the ratio of their areas is....

(iv) If

$$\begin{vmatrix} 3 & \alpha^2 + \beta^2 + \gamma^2 & \alpha + \beta + \gamma \\ \alpha + \beta + \gamma & \alpha^3 + \beta^3 + \gamma^3 & \alpha^2 + \beta^2 + \gamma^2 \\ \alpha^2 + \beta^2 + \gamma^2 & \alpha^4 + \beta^4 + \gamma^4 & \alpha^3 + \beta^3 + \gamma^3 \end{vmatrix} \neq 0,$$

then the points whose rectangular Cartesian co-ordinates are $(1, \alpha, \alpha^2)$, $(1, \beta, \beta^2)$ and $(1, \gamma, \gamma^2)$ are.....

(v) Let a function f be defined by

$$f(x) = \lim_{n \rightarrow \infty} \frac{\ln(2+x) - x^{2n} \sin x}{1+x^{2n}}; \text{ then}$$

$$f(x) = \begin{cases} \dots\dots\dots & \text{when } 0 \leq x < 1 \\ (\ln 3 - \sin 1)/2 & \text{when } x = 1 \\ \dots\dots\dots & \text{when } 1 < x \leq \pi/2. \end{cases}$$

Also f is at $x = 1$.

(vi) The line, $x + y = a$, meets the axes of x and y at A and B respectively. A triangle AMN is

inscribed in the triangle OAB (O , the origin) with right angle at N , and M, N lie respectively on OB and AB . If the area of the triangle AMN is $3/8$ of the area of triangle OAB ; then $AN/NB = \dots\dots\dots$

(vii) The function f defined by $f(x) = x(\ln x)^{-1}$ increases in and decreases in.....

(viii) If $\vec{a} = (0, 1, -1)$ and $\vec{c} = (1, 1, 1)$ are given vectors, then a vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} + \vec{c} = 0$ and $\vec{a} \cdot \vec{b} = 3$ is.....

(ix) The area of the triangle in the Argand plane formed by the complex numbers, z , $-iz$ and $z - iz$ is

(x) If $\alpha, \beta, \gamma, \delta$ are in A.P. and $\int_0^2 f(x) dx = -4$, where

$$f(x) = \begin{vmatrix} x+\alpha & x+\beta & x+\alpha-\gamma \\ x+\beta & x+\gamma & x-1 \\ x+\gamma & x+\delta & x-\beta+\delta \end{vmatrix};$$

then the common difference $d = \dots\dots\dots$

PART B

Note : Justify your answers with mathematical arguments for each question in this part.

3.(a) There are two bags each containing m balls. A man has to select equal number of balls from both the bags. Find the number of ways in which he can do so if he must choose at least one ball from each bag. [3]

(b) Find the co-ordinates of the vertices of a square inscribed in the triangle with vertices $A(0, 0)$, $B(2, 1)$, $C(3, 0)$; given that two of its vertices are on the side AC . [3]

4.(a) Show that $(n+1)^n \leq n^{n+1}$ for all $n \geq 3$. [3]

(b) Find the set of all x satisfying

$$(x-3)\sqrt{x^2+1} > x^2+3$$

[3]

5.(a) The common chord of two intersecting circles can be seen from their centres at the angles of $\pi/2$ and $\pi/3$. Find the radii of the circles if the

distance between their centres is equal to $1 + \sqrt{3}$. [3]

(b) If a function f is with domain and co-domain, the set of rational numbers, such that $f(x+y) = f(x) + f(y)$ for all rationals x and y , then $f(x) = kx$ for all x , where k is some fixed rational number. [3]

6.(a) A box contains 50 tickets numbered 1, 2, ..., 50. Two tickets are chosen at random. It is given that the maximum number on the tickets chosen is not more than 8. Find the probability the minimum such number on them is 4. [2]

(b) Sum of the digits of seven digit number is 59. Find the probability that this number is divisible by 11. [4]

7.(a) OA is a crank 2 metres long which rotates about O; AB is a connecting rod to B which moves upon a straight line passing through O. If α and β be the angles that the crank OA makes with OB, when B has described respectively $\frac{1}{4}$ and $\frac{3}{4}$ of its total travel from its extreme position, then find the value of $\cos(\alpha - \beta)$, the length of AB being 5 metres. [3]

(b) Find the value of x for which the sixth term

of the expansion $\left[2^{\log_2 \sqrt{9^{n-1}+7}} + \frac{1}{2^5 \log_2 (3^{x-1}+1)} \right]^7$

is equal to 84.

8.(a) Let A, B, C be the angles of a triangle such that $A \geq B \geq C$. Find the minimum value of

$$\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} \quad [4]$$

(b) Find all the functions that coincide with their derivatives everywhere. [2]

9.(a) Four distinct points with rectangular Cartesian co-ordinates (0, 0, 0) and $(1, t_r, 2at_r + at_r^3)$, $r = 1, 2, 3$ ($a \neq 0$) are given to be coplanar. Find the value of $t_1 + t_2 + t_3$. [3]

(b). Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$, which have two distinct real roots in the open interval, $0 < x < 1$. Find the least positive integer 'a' for which such a quadratic equation exists. [3]

10. What value should be assigned to $f(0)$ in order that the function f given by $f(x) = \ln(1 + \sin^2 x) \cot \ln^2(1+x)$, $x \neq 0$ is continuous at $x = 0$.

11. Evaluate

(a) $\int_0^{\pi/2} \sin 2x \arctan(\sin x) dx$

(b) $\int_1^6 \arctan(\sqrt{x} - 1) dx$. [6]

12. A normal to the curve, $x^2 + \alpha x - y + 2 = 0$ at the point whose abscissa 1, is parallel to the line, $y = x$. Find the area in the first quadrant bounded by the curve, this normal and the axis of x . [6]

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IIT-JEE '97

PRACTICE TEST SERIES-III

SOLUTIONS

PART A

1.(i) Let O be the centre of pentagon then

$$\angle A_1OA_2 = \angle A_2OA_3 = \angle A_3OA_4 = \angle A_4OA_5$$

$$A_5OA_1 = \frac{360^\circ}{5}$$

$$= 72^\circ \text{ and } r = 1$$

In ΔA_1OA_2 by cosine rule

$$\begin{aligned} &= (A_1A_2)^2 \\ &= 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos 72^\circ \\ &= 2 - 2\cos 72^\circ \end{aligned}$$

In ΔA_1OA_3 by cosine rule $= (A_1A_3)^2$

$$\begin{aligned} &= 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos 144^\circ \\ &= 2 - 2\cos 144^\circ \end{aligned}$$

$$\begin{aligned} \therefore (A_1A_2)(A_1A_3) &= \sqrt{2 - 2\cos 72^\circ} \cdot \sqrt{2 - 2\cos 144^\circ} \\ &= 2\sqrt{1 - \cos 18^\circ} \cdot \sqrt{1 - \cos 36^\circ} = 5. \end{aligned}$$

$$(ii) \quad F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$$\therefore F'(a) = 0$$

$$(\because f_r(a) = h_r(a) = g_r(a) \quad \therefore r = 1, 2, 3)$$

(iii) From given $x = -1, x = 1/3$ are the roots of the equation $f'(x) = 0$

Also notice that $f(x)$ is a 3rd degree function.



$$\therefore f'(x) = \lambda(x+1)(3x-1) \text{ and } f(-2) = 0 \text{ (given)}$$

$$\therefore f'(x) = \lambda(3x^2 + 2x - 1)$$

$$\therefore f(x) = \lambda(x^3 + x^2 - x) + C \quad (\because \text{Integrate})$$

$$f(-2) = \lambda(-8 + 4 + 2) + C = 0 \Rightarrow C = 2\lambda \quad \dots(1)$$

$$\& \int_{-1}^1 f(x) dx = \frac{14}{3}$$

$$\Rightarrow \int_{-1}^1 [\lambda(x^3 + x^2 - x) + C] dx = \frac{14}{3}$$

$$\Rightarrow \lambda + 3C = 7$$

$$\dots(2)$$

($\because x^3, x$ are odd functions)

solve (1) and (2) $\lambda = 1, C = 2$

$$\therefore f(x) = x^3 + x^2 - x + 2.$$

$$(iv) \text{ Given line } \vec{r} = \vec{a} + \lambda \vec{b}$$

$$\dots(1)$$

$$\vec{r} \cdot \vec{n} = 0$$

$$\dots(2)$$

$$\text{Sub. (1) in (2)} = (\vec{a} + \lambda \vec{b}) \cdot \vec{n} = 0$$

$$\vec{a} \cdot \vec{n} + (\vec{b} \cdot \vec{n})\lambda = 0$$

$$\therefore \lambda = -\frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}} \text{ sub. in (1)}$$

$$\therefore \vec{r} = \vec{a} - \left(\frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\right) \cdot \vec{b}.$$

(v) Since the semi latus rectum of a parabola is the harmonic mean between the segments of any focal chord of a parabola therefore SP, 4, SQ are in H.P.

$$4 = \frac{2(SP)(SQ)}{SP + SQ} = \frac{12(SQ)}{6 + SQ} \quad \therefore SQ = 3$$

2. (i). Let A, B, C be the events that the student is successful in tests I, II, III respectively. Then the probability that the student is successful is $p(A \cap B \cap \bar{C}) + p(A \cap \bar{B} \cap \bar{C})$

$$+ p(A \cap B \cap C) = \frac{1}{2}$$

$$pq(1 - \frac{1}{2}) + p(1 - q)\frac{1}{2} + pq\frac{1}{2} = \frac{1}{2}$$

$p(1 + q) = 1$. This is satisfied by $p = 1$; $q = 0$
i.e. (c).

$$\text{Let } p = \frac{n}{n+1} ; q = \frac{1}{n}$$

$\Rightarrow p(1 + q) = 1$ where $n \in \mathbb{N}$.

There are infinite values of p and q i.e. (d)

$\therefore$ Answers (c) and (d).

$$\text{(ii) } \lim_{x \rightarrow 1^-} f(x) = 2; \lim_{x \rightarrow 1^+} f(x) = 2$$

$\therefore f(x)$ is continuous at $x = 1$ i.e. (a)

$$\lim_{x \rightarrow 3^-} f(x) = 0; \lim_{x \rightarrow 3^+} f(x) = 0$$

$\therefore f(x)$ is continuous at $x = 3$ i.e. (c)

$$\text{LHD at } x = 1 \Rightarrow \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = -1$$

$$\text{RHD at } x = 1 \Rightarrow \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = -1$$

$\therefore$ Answers (a), (b), (c).

(iii) The number of required solutions = coefficient of y^n in $(1 + y + y^2 + \dots)^3$

= coefficient of y^n in $(1 - y)^{-3}$

$$= \frac{3 \cdot 4 \cdot 5 \dots (n+2)}{n!} = \frac{(n+2)!}{2! n!}$$

$$= {}^{(n+2)}C_n \text{ i.e. (a)}$$

$$\text{and } {}^{(n+2)}C_n = {}^{(n+2)}C_2 \text{ i.e. (c)}$$

$$(\because {}^nC_r = {}^nC_{n-r})$$

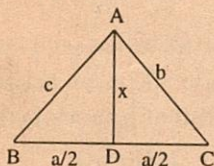
$$\text{and } \frac{(n+2)!}{2! \cdot n!} = \frac{(n+2)(n+1)}{2}$$

$$= \sum (n+1) \text{ i.e. (d)}$$

$\therefore$ Answers (a), (c), (d)

(iv) In $\triangle ABD$

$$x^2 = c^2 + \frac{a^2}{4} - ac \cos B$$



$$4x^2 = 4c^2 + a^2 - 4ac \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$4x^2 = 2(b^2 + c^2) - a^2 \dots (1) \quad \therefore \text{Ans. (a)}$$

$$(1) \Rightarrow 4x^2 = b^2 + c^2 + (b^2 + c^2 - a^2)$$

$$4x^2 = b^2 + c^2 + 2bc \cos A \dots (2) \quad \text{Ans. (b)}$$

$$(1) \Rightarrow 4x^2 = a^2 + 2(b^2 + c^2 - a^2)$$

$$4x^2 = a^2 + 2(2bc \cos A)$$

$$4x^2 = a^2 + 4bc \cos A \dots (3)$$

$\therefore$ Ans (c)

$$(2) \Rightarrow 4x^2 = (b + c)^2 - 2bc(1 - \cos A)$$

$$4x^2 = (b + c)^2 - 4bc \sin^2 \frac{A}{2} \dots (4) \quad \therefore \text{Ans. (d)}$$

$\therefore$ Answers (a), (b), (c), (d).

(v) Let d be the common difference then

$$\log_y x = 1 + d \Rightarrow x = y_1 + d$$

$$\log_z y = 1 + 2d \Rightarrow y = z_1 + 2d$$

$$-15 \log_x z = 1 + 3d \Rightarrow z = x^{-(1+3d)/15}$$

$$\therefore (1+d)(1+2d)(1+3d) + 15 = 0$$

$$d = -2 \quad (\because \text{other two values are imaginary})$$

$$x = y^{-1}; y = z^{-3}; x = z^3 \quad \therefore \text{Answers (a), (b), (c).}$$

$$\begin{aligned} 3. \text{ (i) } \int_2^4 [3 - f(x)] dx &= 7 \Rightarrow \int_2^4 3 dx - \int_2^4 f(x) dx = 7 \\ &\Rightarrow 6 - \int_2^4 f(x) dx = 7 \\ &\Rightarrow \int_2^4 f(x) dx = -1 \end{aligned} \dots (1)$$

$$\begin{aligned} \therefore \int_2^4 f(x) dx &= \int_{-1}^2 f(x) dx - \left[\int_2^4 f(x) dx + \int_4^2 f(x) dx \right] \\ &= - \left[\int_{-1}^4 f(x) dx - \int_2^4 f(x) dx \right] \\ &= - [4 + 1] = 5. \end{aligned} \quad \therefore \text{Ans (c)}$$

$$\text{(ii) } s = \sqrt{at^2 + bt + c}$$

$$v = \frac{ds}{dt} = \frac{2at + b}{2\sqrt{at^2 + bt + c}} \Rightarrow a = \frac{dv}{dt}$$

$$= \frac{1}{2} \left[\frac{2a\sqrt{at^2 + bt + c} - (2at + b)(2at + b)}{2\sqrt{at^2 + bt + c}} \right]$$

$$a = \frac{4a(at^2 + bt + c) - (2at + b)^2}{4(at^2 + bt + c)^{3/2}}$$

$$a = \frac{4ac - b^2}{4s^3} \Rightarrow a \propto \frac{1}{s^3}$$

$\therefore$ Ans (b)

(iii) Let the height of the cylinder be h

$$\text{radius} = r \Rightarrow a^2 = h^2 + r^2$$

$$\text{volume of the cylinder } v = \pi r^2 h$$

$$v = \pi(a^2 - h^2)h$$

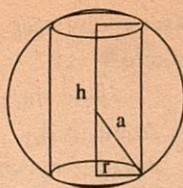
$$\frac{dv}{dh} = \pi(a^2 - 3h^2) = 0$$

$$\Rightarrow h = \pm \frac{a}{\sqrt{3}}$$

$$\text{But } h \text{ cannot be -ve } \therefore h = \frac{a}{\sqrt{3}}$$

$$\therefore \text{Height of the cylinder} = 2h$$

$$= \frac{2a}{\sqrt{3}}$$



Ans. (c)

$$\text{(iv) Equation of the tangent is } S_1^2 = S \cdot S_n$$

$$\Rightarrow (18x + 48y - 144)^2 = (9x^2 + 16y^2 - 144)(36)$$

$$\Rightarrow y^2 + xy - 3x - 8y + 15 = 0$$

$$\Rightarrow y(x + y) - 3(x + y) - 5(y - 3) = 0$$

$$\Rightarrow (y - 3)(x + y - 5) = 0 \quad \therefore y - 3 = 0$$

$$x + y - 5 = 0 \text{ are the equations of the tangents}$$

Ans. (b)

$$\text{(v) } H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1; A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2};$$

$$C \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 \quad \therefore H + C = 2A$$

$\therefore H, A, C$ are in A.P.

$\therefore$ Ans (b)

$$4. \text{ Let } \tan \frac{3\pi}{11} + 4\sin \frac{2\pi}{11} = k$$

$$\frac{3\tan \frac{\pi}{11} - \tan^3 \frac{\pi}{11}}{1 - 3\tan^2 \frac{\pi}{11}} + \frac{8\tan \frac{\pi}{11}}{1 + \tan^2 \frac{\pi}{11}} = k$$

$$\frac{3x - x^3}{1 - 3x^2} + \frac{8x}{1 + x^2} = k \text{ (where } x = \tan \frac{\pi}{11})$$

S.O.B.S

$$(11x - 22x^3 - x^5)^2 = k^2(1 - 2x^2 - 3x^4)^2 \quad \dots(1)$$

Again forming the equation whose roots are

$$\pm \tan \frac{\pi}{11}, \pm \tan \frac{2\pi}{11}, \pm \tan \frac{3\pi}{11}, \pm \tan \frac{4\pi}{11}, \pm \tan \frac{5\pi}{11},$$

$$\text{it is } x^{10} - 55x^8 + 330x^6 - 462x^4 + 165x^2 - 11 = 0$$

$$\Rightarrow (11x - 22x^3 - x^5)^2 = 11(1 - 2x^2 - 3x^4)^2 \quad \dots(2)$$

compare (1) and (2)

$$\therefore k^2 = 11 \Rightarrow \sqrt{11}$$

$$\therefore \tan \frac{3\pi}{11} + 4\sin \frac{2\pi}{11} = \sqrt{11}$$

$$5. \sin x + \sin y = \sqrt{2}$$

$$\Rightarrow \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow uv = \frac{1}{\sqrt{2}} \quad \dots(1)$$

$$\text{(where } u = \sin\left(\frac{x+y}{2}\right); v = \cos\left(\frac{x-y}{2}\right)$$

$$\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)]$$

$$\frac{1}{2} = \frac{1}{2} \left[1 - 2\sin^2\left(\frac{x+y}{2}\right) + 2\cos^2\left(\frac{x-y}{2}\right) - 1 \right]$$

$$\therefore \cos^2\left(\frac{x-y}{2}\right) - \sin^2\left(\frac{x+y}{2}\right) = \frac{1}{2}$$

$$\Rightarrow v^2 - u^2 = \frac{1}{2} \quad \dots(2)$$

$$\text{Solve (1) and (2); } u = \frac{1}{\sqrt{2}}$$

$$\Rightarrow v = 1 \text{ and } u = -\frac{1}{\sqrt{2}} \Rightarrow v = -1$$

$$\sin\left(\frac{x+y}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{x+y}{2} = m\pi + (-1)^m \frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{2} = 2n\pi$$

$$\cos\left(\frac{x-y}{2}\right) = 1$$

$$\Rightarrow \frac{x-y}{2} = 2n\pi$$

$$\sin\left(\frac{x+y}{2}\right) = -\frac{1}{\sqrt{2}}$$

$$\frac{x+y}{2} = m\pi + (-1)^{m+1} \frac{\pi}{4}$$

$$\cos\left(\frac{x+y}{2}\right) = -1$$

$$\Rightarrow \frac{x-y}{2} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \begin{aligned} x &= \pi(m+2n) + (-1)^m \frac{\pi}{4} \\ y &= \pi(m-2n) + (-1)^m \frac{\pi}{4} \end{aligned}$$

$$\Rightarrow \begin{aligned} x &= \pi(m+2n+1) + (-1)^m \frac{\pi}{4} \\ y &= \pi(m-2n-1) + (-1)^{m+1} \frac{\pi}{4} \end{aligned}$$

$$6. \text{ Given curve } 2x^2 + y^2 = 1 \quad \dots(1)$$

$$\text{Given line } 2x + y - 2 = 0 \quad \dots(2)$$

$$\text{Let any point on (1) be } P(x_1, y_1)$$

$$\therefore 2x_1^2 + y_1^2 = 1 \quad \dots(3)$$

$$\text{Let any point on the locus of the}$$

$$\text{image of (1) be } Q(x_2, y_2)$$

$$\therefore \text{By Image concept :}$$

$$\left(x_1 - 4\left(\frac{2x_1 + y_1 - 2}{5}\right), y_1 - 2\left(\frac{2x_1 + y_1 - 2}{5}\right)\right) = (x_2, y_2)$$

$$\left(\frac{-3x_1 - 4y_1 + 8}{5}, \frac{-4x_1 + 3y_1 + 4}{5}\right) = (x_2, y_2)$$

$$2x^2 + y^2 = 1$$

$$\frac{-3x_1 - 4y_1 + 8}{5} = x_2;$$

$$\frac{-4x_1 + 3y_1 + 4}{5} = y_2$$

By symmetry : $\frac{-3x_2 - 4y_2 + 8}{5} = x_1;$

$$\frac{-4x_2 + 3y_2 + 4}{5} = y_1 \text{ sub. in (3)}$$

$$\text{i.e. } 2x_1^2 + y_1^2 = 1$$

$$2(-3x_2 - 4y_2 + 8)^2 + (-4x_2 + 3y_2 + 4)^2 = 25$$

Locus of (x_2, y_2) is

$$2(3x + 4y - 8)^2 + (4x - 3y - 4)^2 = 25$$

7. Let A, B, C be the centres of the three circles

so that $AB = a + b$; $BC = b + c$; $CA = a + c$

Let O be the centre of the circle and λ its radius

$\therefore OA = a + \lambda$;

$OB = b + \lambda$; $OC = c + \lambda$

Let $\angle BOC = 2\alpha$; $\angle COA = 2\beta$; $\angle AOB = 2\gamma$

$$\text{From the } \triangle BOC = \sin^2 \alpha = \sin^2 \left(\frac{\angle BOC}{2} \right)$$

$$= \frac{bc}{(b + \lambda)(c + \lambda)}$$

Similarly $\sin^2 \beta$ and $\sin^2 \gamma$.

And $\alpha + \beta + \gamma = \pi$

$$\therefore (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2)^2$$

$$= 4(1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \gamma)$$

$$(\sum \sin^4 \alpha - 2 \sum \sin^2 \alpha \sin^2 \beta + 4 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma) = 0$$

$$\sum b^2 c^2 (a + \lambda)^2 - 2abc \sum a(b + \lambda)(c + \lambda)$$

$$+ 4a^2 b^2 c^2 = 0$$

$$\lambda^2 [\sum b^2 c^2 - 2abc \sum a] + \lambda [2abc \sum bc - 4abc \sum bc]$$

$$+ a^2 b^2 c^2 = 0$$

$$\lambda^2 \left[2 \sum \left(\frac{1}{bc} \right) - \sum \left(\frac{1}{a^2} \right) \right] + 2\lambda \sum \left(\frac{1}{a} \right) - 1 = 0$$

8. The normal at the point of the eccentric angle ϕ is

$a \sec \phi$ - by $\csc \phi = a^2 - b^2$

(where $\phi = \theta_1, \theta_2, \theta_3, \theta_4$)

It is passing through

the point (h, k) .

$\therefore a h \sec \phi - b y \csc \phi = c^2$

(where $c^2 = a^2 - b^2$)

changing into purely $\cos \phi$ and $\sin \phi$ respectively.

$$\therefore c^4 \cos^4 \phi - 2c^2 a h \cos^3 \phi + (a^2 h^2 + b^2 k^2 - c^4) \cos^2 \phi$$

$$+ 2c^2 a h \cos \phi - a^2 h^2 = 0$$

$$\text{and } c^4 \sin^4 \phi + 2c^2 b k \sin^3 \phi + (a^2 h^2 + b^2 k^2 - c^4) \sin^2 \phi$$

$$- 2c^2 b k \sin \phi - b^2 k^2 = 0$$

$\theta_1, \theta_2, \theta_3, \theta_4$ are the solutions and

$$\sum \cos \phi_1 \cdot \cos \phi_2 = \sum \sin \phi_1 \cdot \sin \phi_2 = 0$$

$$\therefore a^2 h^2 + b^2 k^2 = c^4$$

$$\therefore \text{Locus is } a^2 x^2 + b^2 y^2 = c^4$$

$$x^2 + y^2 \cdot \frac{b^2}{a^2} = \frac{c^4}{a^2}$$

$$x^2 + y^2 \frac{a^2(1 - e^2)}{a^2} = \frac{(a^2 - b^2)^2}{a^2} \quad (\because c^2 = a^2 - b^2)$$

$$x^2 + y^2(1 - e^2) = a^2 e^4 \quad (\because b^2 = a^2(1 - e^2))$$

9. Let $y = mx + n$ be the common asymptote. The

x -co-ordinate of the points of intersection of

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and

$y = mx + n$ are given by

$$x^2(a + 2hm + bm^2) + 2x(hn + bmn + g + fm) + (bn^2 + 2fn + c) = 0$$

$y = mx + n$ is an asymptote to the conic

$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ then

$$x^2(a' + 2h'm + b'm^2) + 2x(h'n + b'mn + g' + f'm) + (b'n^2 + 2f'n + c') = 0$$

$$\text{Then } a + 2hm + bm^2 = 0 \quad \dots(1)$$

$$\text{and } hn + bmn + g + fm = 0 \quad \dots(2)$$

$$a' + 2h'm + b'm^2 = 0 \quad \dots(3)$$

$$\text{and } h'n + b'mn + g' + f'm = 0 \quad \dots(4)$$

Eliminate m from (1) and (3)

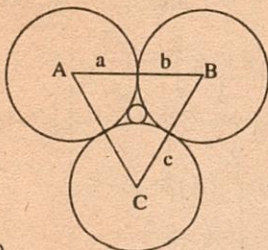
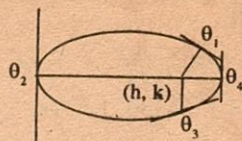
$$\begin{array}{ccccc} 2h & & b & & a \\ & \backslash & & / & \\ 2h' & & b' & & a' \end{array}$$

$$\frac{m^2}{2hb' - 2h'b} = \frac{m}{a'b - ab'} = \frac{1}{2h'a - 2ha}$$

The required condition is $H^2 = 4AB$

where $A = bh' - b'h$; $B = ha' - h'a$

$H = ab' - a'b$



$$10.(i) h(x) = [f(x)]^2 + [g(x)]^2$$

$$\therefore h'(x) = 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)$$

$$h'(x) = -2f''(x)f'(x) + 2f'(x)f''(x)$$

$$\therefore h'(x) = 0$$

$$\therefore \int h'(x) dx = \int 0 dx$$

$$\therefore h(x) = \text{constant.}$$

$$\text{But G.T } h(3) = 5 \Rightarrow h(6) = 5$$

$$(ii) f'(5) = \lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h}$$

$$f'(5) = \lim_{h \rightarrow 0} \frac{f(h) - f(5) - f(0)}{h} \quad (\because f(5) = f(5+0) = f(5) \cdot f(0))$$

$$f'(5) = f(5) \cdot \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ = f(5) \times f'(0) = 2 \times 3 = 6.$$

$$11. \text{ We know that } \int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{x-x^2}}$$

$$= \cos^{-1}(1-2x) = \text{vers}^{-1}(2x)$$

$$\text{But given integral is } \int \frac{(\alpha + \beta x + \gamma x^2 + \delta x^3)}{\sqrt{x(1-x)}} dx$$

$$= \alpha \int \frac{dx}{\sqrt{x-x^2}} + \frac{\beta}{2} \int \frac{\{1-(1-2x)\}}{\sqrt{x-x^2}} dx \\ + \gamma \int \frac{x-x(1-x)}{\sqrt{x^2-x}} dx + \delta \int \frac{x-x(1-x)-x^2(1-x)}{\sqrt{x-x^2}} dx \\ = \alpha \text{vers}^{-1}(2x) + \frac{\beta}{2} \text{vers}^{-1}(2x) - \beta \sqrt{x-x^2} + \\ - \frac{\gamma}{2} \text{vers}^{-1}(2x) - \gamma \sqrt{x-x^2} +$$

$$\gamma \left\{ \frac{(1-2x)\sqrt{x-x^2}}{4} + \frac{1}{8} \sin^{-1}(1-2x) \right\} + \frac{\delta}{2} \text{vers}^{-1}(2x) \\ - \delta \sqrt{x-x^2} + \frac{3\delta}{2} \left\{ \frac{(1-2x)\sqrt{1-x^2}}{4} + \frac{1}{8} \sin^{-1}(1-2x) \right\} + \\ \frac{\delta}{2} (x-x^2)^{3/2}$$

If this integral is independent of circular function then

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\gamma}{8} + \frac{\gamma}{2} - \frac{3\delta}{16} = 0$$

$$\therefore 16\alpha + 8\beta + 8\gamma + 2\gamma + 8\delta - 3\delta = 0$$

$$\therefore 16\alpha + 8\beta + 10\gamma + 5\delta = 0$$

$$12. y = \sqrt{4-x^2} \quad \dots(1)$$

$$x^2 + y^2 - 4x = 0 \dots(2)$$

Solve (1) and (2)

$$\Rightarrow 4 - 4x = 0$$

$$\Rightarrow x = 1 \Rightarrow y = \sqrt{3}$$

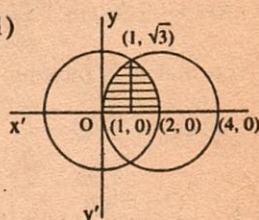
in the first quadrant

$$\text{Required area} = 2 \int_1^2 \sqrt{4-x^2} dx$$

$$= 2 \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} \right]_1^2$$

$$= 2 \left[\left(1 \times 0 + 2 \times \frac{\pi}{2} \right) - \left(\frac{1}{2} \sqrt{3} + 2 \times \frac{\pi}{6} \right) \right]$$

$$= 2 \left[\pi - \frac{\sqrt{3}}{2} - \frac{\pi}{3} \right] = \frac{4\pi - 3\sqrt{3}}{3} \text{ sq. units}$$



13. Let the slopes of the tangent of required family

$$\text{be } \frac{dy}{dx} = m_1.$$

$$\text{G.T. } xy = c \Rightarrow y = c/x$$

$$\therefore \frac{dy}{dx} = \frac{-c}{x^2} = m_2 \text{ (say)}$$

But the angle between the tangents

$$\text{of the curves} = \frac{\pi}{4}$$

$$\therefore \tan \frac{\pi}{4} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$1 = \frac{\frac{dy}{dx} + \frac{c}{x^2}}{1 - \frac{dy}{dx} \cdot \frac{c}{x^2}}$$

$$\therefore \frac{dy}{dx} + \frac{c}{x^2} = 1 - \frac{dy}{dx} \cdot \frac{c}{x^2}$$

$$\left(1 + \frac{c}{x^2} \right) \frac{dy}{dx} = 1 - \frac{c}{x^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - c}{x^2 + c} = 1 - \frac{2c}{x^2 + c}$$

$$\int \left(\frac{dy}{dx} \right) dx = \int \left(1 - \frac{2c}{x^2 + c} \right) dx$$

$$y + k = x - 2\sqrt{c} \tan^{-1} \frac{x}{\sqrt{c}} \dots (1)$$

given that it is passing through $(\sqrt{c}, 0) \Rightarrow 0 + k$

$$= \sqrt{c} - 2\sqrt{c} \times \frac{\pi}{4}$$

$$\therefore k = \frac{(2-\pi)\sqrt{c}}{2} \text{ sub in (1)}$$

$$\therefore 2y + (2-\pi)\sqrt{c} = 2x - 4\sqrt{c} \tan^{-1} \frac{x}{\sqrt{c}}$$

$$14. y(1 + e^{x/y})dx + e^{x/y}(y - x)dy = 0$$

$$(1 + e^{x/y})dx + e^{x/y}(1 - (x/y))dy = 0$$

$$\therefore \frac{dx}{dy} + \frac{e^{x/y}}{1 + e^{x/y}} \left(1 - \frac{x}{y} \right) = 0 \quad \text{————— (1)}$$

$$\text{put } x = vy \Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy}$$

($\because$ Given D.E is as function of x/y)
substitute this in (1)

$$\therefore v + y \frac{dv}{dy} + \frac{e^v}{1 + e^v} (1 - v) = 0$$

$$\frac{1 + e^v}{v + e^v} dv + \frac{dy}{y} = 0 \text{ (V.S.F)}$$

$$\therefore \log(v + e^v) + \log y = \log k \quad (\because \text{By Integrating})$$

$$y(v + e^v) = k \Rightarrow y \left(\frac{x}{y} + e^{x/y} \right) = k$$

$\Rightarrow x + ye^{x/y} = k$ is the required solution

15. Let the required least number of bombs be n
 $p = P$ (hitting the bridge)

$$= \frac{1}{2} \Rightarrow q = \frac{1}{2} \quad (\because p + q = 1)$$

Let x be the random variable showing the number of times the bridge is hit then $p(x \geq 2) > 0.9$

$$= [1 - p(x < 2)] > 0.9 = 1 - 0.9 > p(x < 2)$$

$$= 0.1 > p(x < 2)$$

$$\Rightarrow [p(x = 0) + p(x = 1)] < 0.1$$

$$\Rightarrow nc_0q^n + nc_1pq^{n-1} < 0.1$$

$$\Rightarrow \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n < 0.1 \Rightarrow 10(n + 1) < 2^n$$

$$\text{If } n = 1 \Rightarrow 20 < 2$$

$$\text{If } n = 2 \Rightarrow 30 < 8$$

$$\text{If } n = 7 \Rightarrow 80 < 2^7 \quad \text{i.e. } 80 < 128 \quad n \geq 7$$

$$16. S_r = \alpha^r + \beta^r + \gamma^r \Rightarrow S_0 = 3; S_1 = \sum \alpha$$

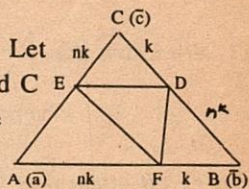
$$S_2 = \sum \alpha^2; S_3 = \sum \alpha^3; S_4 = \sum \alpha^4$$

$$\therefore \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \begin{vmatrix} 3 & \sum \alpha & \sum \alpha^2 \\ \sum \alpha & \sum \alpha^2 & \sum \alpha^3 \\ \sum \alpha^2 & \sum \alpha^3 & \sum \alpha^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}^2 = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2$$

17. Taking A as the origin. Let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The position vectors of F, D, E are respectively:



$$\frac{n\vec{b}}{n+1}, \frac{n\vec{c} + \vec{b}}{n+1}, \frac{\vec{c}}{n+1} \quad \therefore \vec{EF} = \vec{AF} - \vec{AE}$$

$$\vec{EF} = \frac{n\vec{b} - \vec{c}}{n+1} \text{ and } \vec{ED} = \vec{AD} - \vec{AE} = \frac{n\vec{c} + \vec{b} - n\vec{b}}{n+1}$$

$$\therefore \text{Vector Area of the } \Delta ABC = \frac{1}{2} (\vec{AB} \times \vec{AC})$$

$$= \frac{1}{2} (\vec{b} \times \vec{c}) \quad \dots \dots (1)$$

$$\therefore \text{Vector area of the } \Delta EFD = \frac{1}{2} (\vec{EF} \times \vec{ED})$$

$$= \frac{1}{2} \left(\frac{n\vec{b} - \vec{c}}{n+1} \times \frac{n\vec{c} + \vec{b} - n\vec{b}}{n+1} \right)$$

$$= \frac{1}{2(n+1)^2} \{ n^2(\vec{b} \times \vec{c}) - (1-n)(\vec{c} \times \vec{b}) \}$$

$$(\because \vec{c} \times \vec{c} = 0, \vec{b} \times \vec{b} = 0)$$

$$= \frac{1}{2(n+1)^2} \{ n^2(\vec{b} \times \vec{c}) + (1-n)(\vec{b} \times \vec{c}) \}$$

$$(\because \vec{c} \times \vec{b} = -(\vec{b} \times \vec{c}))$$

$$= \frac{1}{2(n+1)^2} (n^2 - n + 1)(\vec{b} \times \vec{c})$$

$$= \frac{(n^2 - n + 1)}{(n+1)^2} \cdot \frac{1}{2} (\vec{b} \times \vec{c})$$

$$= \frac{n^2 - n + 1}{(n+1)^2} \cdot \Delta ABC$$

IIT-JEE '96 SOLVED PAPER

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1. There are 4 parts in this question. Each part has exactly one correct answer. Indicate the correct answer for each part by writing one of the letters A, B, C or D in the answer book.

(i) For the three events A, B and C, P (exactly one of the events A or B occurs) = P (exactly one of the events B or C occurs) = P (exactly one of the events C or A occurs) = p and P (all the three events occur simultaneously) = p^2 , where $0 < p < \frac{1}{2}$. Then the probability of at least one of the three events A, B and C occurring is

(a) $\frac{3p+2p^2}{2}$

(b) $\frac{p+3p^2}{4}$

(c) $\frac{p+3p^2}{2}$

(d) $\frac{3p+2p^2}{4}$

Soln.: According to the question

$$P(A) + P(B) - 2P(A \cap B) = p \quad \dots(i)$$

$$P(B) + P(C) - 2P(B \cap C) = p \quad \dots(ii)$$

$$P(C) + P(A) - 2P(C \cap A) = p \quad \dots(iii)$$

$$P(A + B + C) = p^2 \quad \dots(iv)$$

Adding (i), (ii), and (iii) we get

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3p}{2} \quad \dots(v)$$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A + B + C)$$

$$= \frac{3p}{2} + p^2$$

Putting the value from equation (iv) and (v) we get

$$P(A \cup B \cup C) = \frac{3p}{2} + p^2 = \frac{3p+2p^2}{2} \quad \therefore \text{Required answer (a)}$$

(ii) The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α . The equation of the locus of the point P is

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$

(b) $x^2 + y^2 + 4x - 6y - 9 = 0$

(c) $x^2 + y^2 + 4x - 6y - 4 = 0$

(d) $x^2 + y^2 + 4x - 6y + 9 = 0$

Soln.: Let AP and PB be two tangents C be the centre of circle

$$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

So coordinates of the centre are $(-2, 3)$

Radius of circle $CB = 2\sin\alpha$

In $\triangle PCB$, $\angle PBC = 90^\circ$

$$\sin\alpha = \frac{CB}{CP}$$

$$= \frac{2\sin\alpha}{CP} \quad \therefore CP = 2$$

Let the coordinates of P be (α, β) then

$$OP^2 = (\alpha + 2)^2 + (\beta - 3)^2$$

$$4 = \alpha^2 + 4\alpha + 4 + \beta^2 - 6\beta + 9$$

$$\alpha^2 + \beta^2 + 4\alpha - 6\beta + 9 = 0$$

So locus of P is $x^2 + y^2 + 4x - 6y + 9 = 0$

Required answer (d)

(iii) $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$

(b) $x = y, x \neq 0$

(c) $x = y$

(d) $x \neq 0, y \neq 0$

Soln.:

$$\text{Since } \sec^2\theta = \frac{4xy}{(x+y)^2}$$

$$\text{so } 1 \geq \sec^2\theta$$

$$\text{But } \sec\theta \in]-1, 1[$$

$$\therefore \sec^2\theta = 1$$

$$\frac{4xy}{(x+y)^2} = 1$$

$$(x+y)^2 - 4xy = 0$$

$$(x-y)^2 = 0$$

$$\therefore x = y \neq 0$$

$$\therefore \sec^2\theta = \frac{4xy}{(x+y)^2} \text{ is true if } x = y, x \neq 0$$

Required answer is (b).

(iv) For positive integers n_1, n_2 the value of the expression

$$(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$$

$$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

where $i = \sqrt{-1}$, is a real number if and only if

- (a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
 (c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$.

Soln.:

Now $(1 + i)^{n_1} + (1 + i^3)^{n_1}$

$$\begin{aligned} &= (1 + i)^{n_1} + (1 - i)^{n_1} \\ &= {}^{n_1}C_0 + {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 + {}^{n_1}C_3 i^3 + \dots + {}^{n_1}C_{n_1} - {}^{n_1}C_1 i \\ &\quad + {}^{n_1}C_2 i^2 - {}^{n_1}C_3 i^3 + \dots \\ &= 2[{}^{n_1}C_0 - {}^{n_1}C_2 + {}^{n_1}C_4 - {}^{n_1}C_6 + \dots] \quad \dots(i) \end{aligned}$$

Again $(1 + i^5)^{n_2} + (1 + i^7)^{n_2}$

$$\begin{aligned} &= (1 + i)^{n_2} + (1 - i)^{n_2} \\ &= 2[{}^{n_2}C_0 - {}^{n_2}C_2 + {}^{n_2}C_4 - {}^{n_2}C_6 + \dots] \quad \dots(ii) \end{aligned}$$

Adding (i) and (ii) we get

$$\begin{aligned} &(1 + i)^{n_1} + (1 + i^3)^{n_1} + (1 + i^5)^{n_2} + (1 + i^7)^{n_2} \\ &= 2[({}^{n_1}C_0 + {}^{n_2}C_0) - ({}^{n_1}C_2 + {}^{n_2}C_2) + ({}^{n_1}C_4 + {}^{n_2}C_4) \dots] \end{aligned}$$

which is real for $n_1 > 0$ and $n_2 > 0$

Required answer (d).

2. Find the intervals of values of "a" for which the line $y + x = 0$ bisects two chords drawn from

a point $\left(\frac{1 + \sqrt{2}a}{2}, \frac{1 - \sqrt{2}a}{2}\right)$ to the circle

$$2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$$

Soln.:

Circle is $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$

$$x^2 + y^2 - \left(\frac{1 + \sqrt{2}a}{2}\right)x - \left(\frac{1 - \sqrt{2}a}{2}\right)y = 0 \quad \dots(i)$$

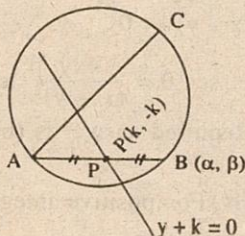
We have

$$\begin{aligned} &\left(\frac{1 + \sqrt{2}a}{2}\right)^2 + \left(\frac{1 - \sqrt{2}a}{2}\right)^2 - \left(\frac{1 + \sqrt{2}a}{2}\right) \cdot \left(\frac{1 + \sqrt{2}a}{2}\right) \\ &\quad - \left(\frac{1 - \sqrt{2}a}{2}\right) \cdot \left(\frac{1 - \sqrt{2}a}{2}\right) = 0 \end{aligned}$$

Hence $A \equiv \left(\frac{1 + \sqrt{2}a}{2}, \frac{1 - \sqrt{2}a}{2}\right)$ is a point on the circle (i)

Let AB and AC be the two chords drawn through A. Let the line $y + x = 0$, bisect the line AB at the point P. Then $P \equiv (k, -k)$

Let $B = (\alpha, \beta)$



$$\therefore \frac{\alpha + \frac{1 + \sqrt{2}a}{2}}{2} = k$$

$$\alpha = 2k - \left(\frac{1 + \sqrt{2}a}{2}\right) \text{ and}$$

$$\text{similarly } \beta = -2k - \left(\frac{1 - \sqrt{2}a}{2}\right)$$

But B is a point on the circle (i)

$$\begin{aligned} &\left[2k - \left(\frac{1 + \sqrt{2}a}{2}\right)\right]^2 + \left[-2k - \left(\frac{1 - \sqrt{2}a}{2}\right)\right]^2 \\ &\quad - \frac{1}{2}[1 + \sqrt{2}a] \times \left[2k - \left(\frac{1 + \sqrt{2}a}{2}\right)\right] \\ &\quad - \frac{1}{2}(1 - \sqrt{2}a) \left[-2k - \frac{1}{2}(1 - \sqrt{2}a)\right] = 0 \end{aligned}$$

$$\text{or } 8k^2 - 6\sqrt{2}ak + 1 + 2a^2 = 0$$

Hence for k real and different we get

$$(6\sqrt{2}a)^2 - 4 \cdot 8(1 + 2a^2) > 0$$

$$\Rightarrow 2a^2 - 32 - 64a^2 > 0$$

$$\Rightarrow a^2 > 4$$

$$\therefore a < -2 \text{ and } a > 2$$

$$\therefore a \in (-\infty, -2) \cup (2, \infty) \text{ Ans.}$$

3. The question contains four incomplete statements. In each part, fill in the blanks so that each of the resulting statement is correct. Write your answers in your answer book in the order in which the statements are given below.

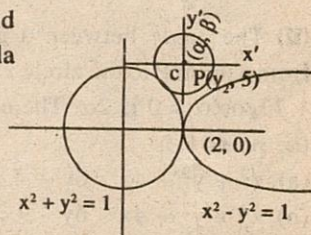
(i) An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P(\frac{1}{2}, 1)$. Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse, in the standard form is

Soln.:

The common tangents to the circle $x^2 + y^2 = 1$ and the rectangular hyperbola $x^2 - y^2 = 1$ is $x = 1$.

This is the directrix of the required ellipse.

Therefore, the major diameter is parallel



to x-axis. Since $P(\frac{1}{2}, 1)$ is a focus of the required ellipse. Its centre can be taken as $(\alpha, 1)$.

We have $CP = (\frac{1}{2} - \alpha) = a \cdot e = a/2$

$\therefore e = \frac{1}{2}$ given and $1 - \alpha = a/e = 2a$

$\therefore \frac{1}{2} = 3/2 a \Rightarrow a = 1/3$

$$\therefore \alpha = 1 - \frac{2}{3} = \frac{1}{3}$$

$$b^2 = a^2(1 - e^2)$$

$$= \frac{1}{9} \left(1 - \frac{1}{4}\right) = \frac{1}{9} \times \frac{3}{4} = \frac{1}{12}$$

Hence the required equation of the ellipse is

$$\frac{(x - 1/3)^2}{(1/3)^2} + \frac{(y - 1)^2}{1/12} = 1$$

$$\therefore 9(x - 1/3)^2 + 12(y - 1)^2 = 1 \text{ Ans.}$$

(ii) If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation

$$f(x) = f\left(\frac{x+1}{x+2}\right) \text{ are } \dots\dots, \dots\dots, \dots\dots \text{ and } \dots\dots$$

Soln.:

$$f(x) = f\left(\frac{x+1}{x+2}\right)$$

For even function $f(x) = f(-x)$

$$\therefore f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x) \quad \therefore f(x) = f\left(\frac{x+1}{x+2}\right)$$

$$x = \frac{x+1}{x+2} \Rightarrow x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2} \quad f(-x) = f\left(\frac{x+1}{x+2}\right)$$

$$-x = \frac{x+1}{x-2} \Rightarrow x^2 + 3x + 1 = 0$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

$$\therefore x = (-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2 \text{ and } (-3 + \sqrt{5})/2.$$

(iii) General value of θ satisfying the equation $\tan^2\theta + \sec^2\theta = 1$ is

$$\text{Soln.: } \tan^2\theta + \sec^2\theta = 1$$

$$\Rightarrow \tan^2\theta + \frac{1 + \tan^2\theta}{1 - \tan^2\theta} = 1$$

$$\Rightarrow \tan^2\theta - \tan^4\theta + 1 + \tan^2\theta = 1 - \tan^2\theta$$

$$\Rightarrow 3\tan^2\theta - \tan^4\theta = 0$$

$$\Rightarrow \tan^2\theta[3 - \tan^2\theta] = 0$$

$$\text{Either } \tan^2\theta = 0$$

$$\text{or } 3 - \tan^2\theta = 0,$$

$$\text{If } \tan^2\theta = 0$$

$$\therefore \theta = n\pi$$

$$\text{Again } 3 - \tan^2\theta = 0$$

$$\tan\theta = \pm\sqrt{3} = \tan(\pm\pi/3)$$

$$\theta = m\pi \pm \pi/3$$

$\therefore \theta = n\pi, m\pi \pm \pi/3$ where m, n are two integer including zero.

(iv) For any odd integer $n \geq 1$,

$$n^3 - (n-1)^3 + \dots + (-1)^{n-1} 1^3 = \dots\dots\dots$$

Soln.:

The given series is

$$\frac{(3n^2 - 3n + 1) + (3(n-2)^2 - 3(n-2) + 1) + \dots + 1}{\frac{n-1}{2} \text{ terms}}$$

$$\text{Take } T_K = 3(2K+1)^2 - 3(2K+1) + 1$$

$$\text{sum} = \sum_{K=1}^{(n-1)/2} T_K$$

$$= \sum_{K=1}^{(n-1)/2} [3(2K+1)^2 - 3(2K+1) + 1]$$

$$= \sum_{K=1}^{(n-1)/2} 12K^2 + 6K + 1 = (2n-1)(n+1)^2.$$

4. A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at P is a $(y-1) + (x-1) = 0$. If the slope of the tangent at any point or the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P .

Soln.: Normal at $P = ay + x = a + 1$

$$\frac{dy}{dx} \Big|_{a+p} = a \quad \dots(1)$$

$$\frac{dy}{dx} = \tan\theta = Ky$$

$$\text{Using (1) } \frac{dy}{dx} = ay$$

on solving $\ln y = ax + c$

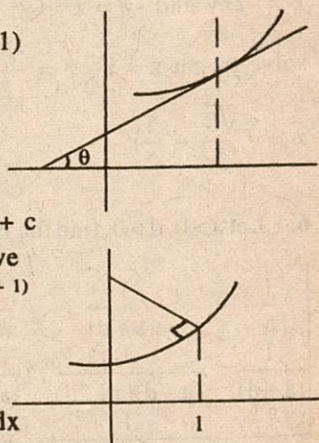
$(1, 1)$ lies on the curve

$$\Rightarrow c = -a \Rightarrow y = e^{a(x-1)}$$

is the required curve

Required Area =

$$\int_0^1 \left[\frac{1-x}{a} + 1 - e^{a(x-1)} \right] dx$$



$$= \left| \frac{x}{a} - \frac{x^2}{2a} + 1 - \frac{1}{a} \right|_0^1$$

$$= \left(\frac{1}{a} - \frac{1}{2a} + 1 - \frac{1}{a} \right) + \frac{e^{-a}}{a} = \frac{2e^{-a} - 1 + 2a}{2a}$$

5.(a) Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C, taken in pairs, intersect at points P, Q and R. Determine the ratio of the areas of the triangles ABC and PQR.

Soln.: Let $A \equiv (at_1^2, 2at_1)$; $B \equiv (at_2^2, 2at_2)$, $C \equiv (at_3^2, 2at_3)$. Intersection of tangents at A, B and C are : $P \equiv (at_1t_2, a(t_1 + t_2))$; $Q \equiv (at_2t_3, a(t_2 + t_3))$; $R \equiv (at_3t_1, a(t_3 + t_1))$

$$\frac{\text{Ar}(\Delta ABC)}{\text{Ar}(\Delta PQR)} = \frac{\frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}}{\frac{1}{2} \begin{vmatrix} at_1t_2 & a(t_1 + t_2) & 1 \\ at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \end{vmatrix}}$$

$$= \frac{2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)} = 2$$

(b) Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. [2]

Soln.: Let $z = x + iy$

$$x - iy = i(x^2 - y^2 + 2xyi)$$

$$x = -2xy \text{ and } -y = x^2 - y^2$$

$$\text{solve to get } z = i, \quad z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

6. Let $a > 0, d > 0$. find the value of the determinant.

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Soln.: Operate $C_2 \rightarrow C_1 - C_2$ and $C_3 \rightarrow C_2 - C_3$

$$\Delta = \frac{1}{d^2} \begin{vmatrix} \frac{1}{a} & \frac{1}{a} - \frac{1}{a+d} & \frac{1}{a+d} - \frac{1}{a+2d} \\ \frac{1}{a+d} & \frac{1}{a+d} - \frac{1}{a+2d} & \frac{1}{a+2d} - \frac{1}{a+3d} \\ \frac{1}{a+2d} & \frac{1}{a+2d} - \frac{1}{a+3d} & \frac{1}{a+3d} - \frac{1}{a+4d} \end{vmatrix}$$

$$\Delta = \frac{1}{d^2} \begin{vmatrix} \frac{1}{a} & \frac{1}{a+d} & \frac{1}{a+2d} \\ \frac{1}{a+d} & \frac{1}{a+2d} & \frac{1}{a+3d} \\ \frac{1}{a+2d} & \frac{1}{a+3d} & \frac{1}{a+4d} \end{vmatrix}$$

$$= \frac{1}{d^2 a(a+d)(a+2d)} \begin{vmatrix} 1 & \frac{a}{a+d} & \frac{a}{a+2d} \\ 1 & \frac{a+d}{a+2d} & \frac{a+2d}{a+3d} \\ 1 & \frac{a+2d}{a+3d} & \frac{a+3d}{a+4d} \end{vmatrix}$$

Operate $R_1 \leftarrow R_1 - R_2$ and $R_2 \leftarrow R_2 - R_3$
 $\Delta =$

$$\frac{1}{d^2 a(a+d)(a+2d)} \begin{vmatrix} 0 & \frac{-d^2}{(a+d)(a+2d)} & \frac{-2d^2}{(a+2d)(a+3d)} \\ 0 & \frac{-d^2}{(a+2d)(a+3d)} & \frac{-2d^2}{(a+3d)(a+4d)} \\ 1 & \frac{a+2d}{a+3d} & \frac{a+3d}{a+4d} \end{vmatrix}$$

Evaluate by C_1

$$\frac{1}{d^2 a(a+d)(a+2d)} \cdot 2d^4 \left(\frac{1}{((a+d)(a+2d)(a+3d)(a+4d))} - \frac{1}{(a+2d)^2(a+3d)^2} \right)$$

$$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

7. Determine the points of maxima and minima of the function

$$f(x) = \frac{1}{8} \ln x - bx + x^2, x > 0,$$

where $b \geq 0$ is a constant.

$$\text{Soln.: } f'(x) = \frac{1}{8x} - b + 2x = 0$$

$$\Rightarrow 16x^2 - 8bx + 1 = 0$$

$$\Rightarrow x = \frac{b \pm \sqrt{b^2 - 1}}{4}$$

For $0 \leq b < 1$ $f'(x) > 0$ for all $x \Rightarrow f(x)$ is an increasing function. No. local max. or local min.

$$\text{For } b > 1 \quad \left[\begin{array}{ccc} f'(x) & + & - & + \\ x & \frac{b - \sqrt{b^2 - 1}}{4} & \frac{b + \sqrt{b^2 - 1}}{4} & \end{array} \right]$$

$$\text{is local max.} \quad \text{local max.} \quad \text{local min.}$$

$$\text{For } b = 1; f'(x) = 16x^2 - 8x + 1 = (4x - 1)^2 = 0$$

$$\Rightarrow x = 1/4$$

$$f''(x) = 2(4x - 1)(4) \Rightarrow f''(1/4) = 0$$

$$f''(1/4) = 0 \text{ and } f'''(1/4) \neq 0 \quad x = 1/4 \text{ is a point of inflexion.}$$

So no local maxima or minima.

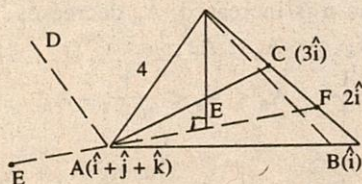
$0 \leq b \leq 1$: - No local max. or min.

$$b > 1 : - \text{local max at } x = \frac{b - \sqrt{b^2 - 1}}{4}$$

$$\text{local min at } x = \frac{b + \sqrt{b^2 - 1}}{4}$$

8. The position vectors of the vertices A, B and C of a tetrahedron ABCD are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the sides AD is 4 and the volume of the tetrahedron is $2\sqrt{2}/3$, find the position vector of the point E for all its possible positions.

Soln.:



$$\text{Volume of tetrahedron} = \frac{\vec{AD} \cdot (\vec{AB} \times \vec{AC})}{6} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \frac{\vec{AD} \cdot \hat{n}}{6} |\vec{AB} \times \vec{AC}| = \frac{2\sqrt{2}}{3} \quad [\hat{n} \text{ is a unit vector } \parallel \text{ to DE}]$$

$$\vec{AB} \times \vec{AC} = -\hat{j} + \hat{k} \times (2\hat{i} - \hat{j} - \hat{k}) = 2\hat{k} - 2\hat{j}$$

$$\Rightarrow \frac{DE}{6} (2\sqrt{2}) = \frac{2\sqrt{2}}{3} \Rightarrow DE = 2$$

$$\text{In rt } \triangle ADE \quad AE = \sqrt{4^2 - 2^2} = 2\sqrt{3}$$

$$\vec{AE} = 2\sqrt{3} \frac{\vec{AF}}{|\vec{AF}|} = 2\sqrt{3} \frac{(\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}} = 2(\hat{i} - \hat{j} - \hat{k})$$

$$\vec{AE} = \vec{e} - \vec{a} \Rightarrow \vec{e} = 2(\hat{i} - \hat{j} - \hat{k}) + \hat{i} + \hat{j} + \hat{k} = 3\hat{i} - \hat{j} - \hat{k}.$$

Ind case When E lies outside the triangle ABC i.e on the other side of median AF, (as shown in the figure)

$$\vec{AE} = -2(\hat{i} - \hat{j} - \hat{k}) \Rightarrow \vec{e} = -\hat{i} + 3\hat{j} + 3\hat{k}.$$

9.(a). The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.

(b) From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola.

Soln.:

$$(a) \text{ Let } x_1 = a - d, x_2 = a, x_3 = a + d$$

$$\text{sum of roots} = 3a = 1 \Rightarrow a = \frac{1}{3}$$

Product of roots taken 2 at a time

$$= a(a - d) + a(a + d) + (a - d)(a + d) = \beta$$

$$= 3a^2 - d^2 = \beta$$

$$\Rightarrow \frac{1}{3} - d^2 = \beta \Rightarrow -\beta + \frac{1}{3} \geq 0 \Rightarrow \beta \leq \frac{1}{3}$$

Product of roots = $a(a - d)(a + d) = -\gamma$

$$\Rightarrow a(a^2 - d^2) = -\gamma \Rightarrow \frac{1}{3} \left(\frac{1}{9} - d^2 \right) = -\gamma$$

$$\Rightarrow \frac{1}{9} + 3\gamma \geq 0 \Rightarrow \gamma \geq -\frac{1}{27}$$

$$\Rightarrow \text{Ans } \beta \leq 1/3 \text{ and } \gamma \geq -1/27$$

$$(b) \text{ Let } A_1 \equiv (at^2, 2at);$$

$$B_1 = \left(\frac{a}{\sqrt{2}} \cos \theta, \frac{a}{\sqrt{2}} \sin \theta \right)$$

Tangent at A_1

$$yt - x = at^2 \dots (1)$$

Tangent at B_1

$$\cos \theta x + \sin \theta y = \frac{a}{\sqrt{2}} \dots (2)$$

equation (1) and (2) are same

$$\Rightarrow -\frac{1}{\cos \theta} = \frac{t}{\sin \theta} = \sqrt{2}t^2 \Rightarrow \frac{1}{2t^4} + \frac{1}{2t^2} = 1$$

(using $\sin^2 \theta + \cos^2 \theta = 1$)

$$\Rightarrow (t^2 - 1)(2t^2 + 1) = 0 \Rightarrow t = \pm 1 \quad \begin{matrix} t = -1 \text{ for } A_1 \\ t = -1 \text{ for } A_2 \end{matrix}$$

$$A_1 A_2 = 4at = 4a$$

$$B_1 B_2 = \sqrt{2}a \sin \theta = \sqrt{2}a \frac{1}{\sqrt{2}} = a$$

$$\left(\sin \theta = \frac{1}{\sqrt{2}t} = \frac{1}{\sqrt{2}} \text{ and } \cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = 135^\circ \right)$$

$$\therefore O_1 O_2 = |OO_1| + |OO_2| = \left| \frac{a}{\sqrt{2}} \cos \theta \right| + |at^2| = \frac{3a}{2}$$

$$\text{Ar}(A_1 B_1 A_2 B_2) = \frac{1}{2}(5a) \frac{3a}{2} = \frac{15a^2}{4}$$

[using : Area(trapezium) = $\frac{1}{2}$ (sum of parallel sides) $\times$ height]

10. A function $f: \mathbb{R} \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is defined by

$$f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$$

Find the interval of values of α for which f is onto.

Is the function one-to-one for $\alpha = 3$? Justify your answer.

Soln.:

$$\text{Let } y = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$$

$$\Rightarrow (\alpha + 8y)x^2 + 6(1 - y)x - 8 - y\alpha = 0$$

As x can take all real values,

$$D > 0 \Rightarrow 36(1 - y)^2 + 4(\alpha + 8y)(8 + y\alpha) \geq 0$$

$$\Rightarrow (9 + 8\alpha)y^2 + (\alpha^2 + 46)y + (9 + 8\alpha) \geq 0 \dots (1)$$

For f to be onto inequation (1) should be true for all values of y

$$\text{i.e. } D \leq 0 \text{ and } 9 + 8\alpha > 0 \dots (2)$$

$$\Rightarrow (\alpha^2 + 46)^2 - 4(9 + 8\alpha)^2 \leq 0$$

$$\Rightarrow (\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0$$

$$\Rightarrow \alpha^2 + 16\alpha + 28 \leq 0$$

$$(\because \alpha^2 + 16\alpha + 64 > 0 \text{ for all } \alpha)$$

$$\Rightarrow (\alpha - 14)(\alpha - 2) \leq 0$$

$$\Rightarrow 2 \leq \alpha \leq 14 \text{ and } \alpha > \frac{9}{8} \text{ (using 2)}$$

$$\Rightarrow 2 \leq \alpha \leq 14$$

One to one means there should be no two values of x for which the value of y is same.

let $\alpha = 3$ and $y = 1$

$$1 = \frac{x^2 + 6x - 8}{3 + 6x - 8x^2} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

so for $x = 1$ and $x = -1$ y has same value i.e. $y = 1$

so function is not one to one for $\alpha = 3$.

11.(a) Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$

$$\text{and deduce } \frac{1}{2n+2} < A_n < \frac{1}{2n-2}.$$

(b) A rectangle PQRS has its side PQ parallel to the line $y = mx$ and vertices P, Q and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R.

Soln.:

$$(a) A_n = \int_0^{\pi/4} (\tan x)^n dx \quad \left[\text{Adding the two.} \right]$$

$$A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2} dx$$

$$A_n + A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2} [1 + \tan^2 x] dx$$

$$\Rightarrow A_n + A_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x = \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4}$$

$$\Rightarrow A_n + A_{n-2} = \frac{1}{n-1}. \text{ Hence proved}$$

As n is increased, $(\tan x)^n$ decreases because

$$0 \leq \tan x \leq 1 \text{ in } x \in [0, \pi/4]$$

$\Rightarrow$ As n is increased, A_n decreases

$$\Rightarrow A_{n-2} < A_n < A_{n+2}$$

$$\Rightarrow \frac{1}{n+1} - A_n < A_n < \frac{1}{n-1} - A_n$$

$$\left[\text{using } A_n + A_{n-2} = \frac{1}{n-1} \right]$$

$$\Rightarrow \frac{1}{n-1} < 2A_n < \frac{1}{n-1}$$

$$\Rightarrow \frac{1}{2n+2} < A_n < \frac{1}{2n+2}$$

(b) Let $P \equiv (t, a)$

Equation of PQ = $y - a$

$$= m(x - t)$$

$Q \equiv (b, a + m(b - t))$

[Put $x = b$ in above equation]

Equation of PS

$$y - a = -\frac{1}{m}(x - t)$$

$$\Rightarrow S \equiv \left[-b, a + \frac{1}{m}(b + t)\right] \text{ Put } x = -b$$

$$\text{Slope of RS} = \frac{y_1 - a - \frac{1}{m}(b + t)}{x_1 + b} = m$$

$$\Rightarrow b + t = m(y_1 - a) - m^2(x + b) \quad \dots(1)$$

$$\text{Slope of RQ} = \frac{y_1 - a - m(b - t)}{x_1 - b} = -\frac{1}{m}$$

$$\Rightarrow \frac{m(y_1 - a) + (x_1 - b)}{m^2} = b - t \quad \dots(2)$$

Add (1) and (2) to eliminate t

$$\Rightarrow 2b = m(y_1 - a) - m^2(x + b) + \frac{m(y_1 - a) + (x_1 - b)}{m^2}$$

$$\Rightarrow \text{locus is } my + (1 - m^2)x - am - b(1 + m^2) = 0$$

12. This question contains three incomplete statements. In each part, fill in the blanks so that each of the resulting statements is correct. Write your answers in your answer book in the order in which the statements are given below.

(i) Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]}\right)$, where $[\cdot]$ denotes the greatest integer function. The domain of f is and the points of discontinuity of f in the domain are....

(ii) The value of the expression $1 \cdot (2 - \omega)(2 - \omega^2) + 2 \cdot (3 - \omega)(3 - \omega^2) + \dots + (n - 1) \cdot (n - \omega)(n - \omega^2)$, where ω is an imaginary cube root of unity, is

(iii) The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is.....

Soln.: (i) $x \in (\infty, -1) \cup (0, \infty)$, $x \in I \cap D$

$$(ii) \left[\frac{n(n+1)}{2}\right]^2 - n$$

$$(iii) x^2 + y^2 - x - y = 0$$

13. This question contains three incomplete statements. In each part, fill in the blanks so that each of the resulting statements is correct. Write your answers in your answer book in the order in which the statements are given below.

(i) Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$. The number of solutions $(x_1, x_2, \dots, x_k)$, $x_1 \geq 1, x_2 \geq 2, \dots, x_k \geq k$, all integers, satisfying $x_1 + x_2 + \dots + x_k = n$, is.....

(ii) If for non-zero x , $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ where $a \neq b$, then $\int_1^2 f(x) dx = \dots$

$$(iii) \lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2}\right)^{\frac{1}{x^2}} = \dots$$

Soln.: $n - 1 - \frac{k(k-1)}{2} C_{k-1}$

$$(ii) \frac{1}{b^2 - a^2} \left[\frac{3b}{2} - a \log 2 - 5b + 5a \right]$$

$$(iii) e^2$$

$$14.(a) \text{ Let } f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$$

where a is a positive constant. find the interval in which $f'(x)$ is increasing.

$$(b) \text{ Evaluate } \int \frac{(x+1)}{x(1+xe^x)^2} dx.$$

Soln.:

$$(a) x \leq 0$$

$$f'(x) = e^{ax}(1 + xa)$$

$$f''(x) = e^{ax}(2a + xa^2) > 0 \Rightarrow x > -2/a$$

($\because e^{ax}$ is always +ve)

$$\text{so } f'(x) \text{ is increasing in } -2/a < x < 0 \quad \dots(1)$$

$$x > 0 \quad f'(x) = 1 + 2ax - 3x^2$$

$$f''(x) = 2a - 6x > 0 \Rightarrow x < a/3$$

$\Rightarrow f'(x)$ is increasing in $0 < x < a/3$

from (1) and (2) $f'(x)$ is increasing in

$$x \in (-2/a, 0) \cup (0, a/3)$$

$$(b) I = \int \frac{x+1}{x(1+xe^x)^2} dx \quad \text{let } 1+xe^x = t$$

$$\Rightarrow e^x(1+x)dx = dt$$

$$\Rightarrow I = \int \frac{dt}{(t-1)t^2} \quad \text{put } t = 1/z \Rightarrow I = \int \frac{zdz}{z-1}$$

$$\Rightarrow I = z + \ln|z-1|$$

$$\Rightarrow I = \frac{1}{1+xe^x} + \ln \left| \frac{xe^x}{1+xe^x} \right| + C.$$

15. Determine the equation of the curve passing through the origin, in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x+6y)$.

Soln.:

$$\text{Let } 10x + 6y = m \Rightarrow \frac{dy}{dx} = \frac{1}{6} \left(\frac{dm}{dx} - 10 \right)$$

$$\text{so we get, } \frac{dm}{dx} = 2(3\sin m + 5)$$

$$\Rightarrow \int \frac{dx}{2(3\sin m + 5)} = \int dx$$

Put $\tan m/2 = y$ and solve integral on LHS

$$\text{We get } \frac{1}{4} \tan^{-1} \left(\frac{5y+3}{4} \right) = x + C$$

$$\therefore \text{ curve passing through } (0, 0) \Rightarrow +1/4 \tan^{-1} 3/4$$

$$\Rightarrow \tan(4x + \tan^{-1} 3/4) = \frac{5 \tan(5x + 3y) + 3}{4}$$

$$\text{Simplify to get } y = \frac{1}{3} \tan^{-1} \left(\frac{5 \tan 4x}{4 - 3 \tan 4x} \right) - \frac{5x}{3}$$

$$\left[\text{use } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \right]$$

16.(i) Using mathematical induction prove that for every integer $n \geq 1$, $(3^{2^n} - 1)$ is divisible by $2n + 2$ but not by 2^{n+3} .

(ii) Find all values of θ in the interval $(-\pi/2, \pi/2)$ satisfying the equation

$$(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2^{\tan \theta} = 0.$$

Soln.:

(a) Let $P(n) = 3^{2^n} - 1$ is divisible by 2^{n+2} , but not divisible by 2^{n+3} .

$P(1) = 8$ is divisible by 2^3 , but not divisible by 2^4
 $\Rightarrow P(1)$ is true.

let $P(k)$ be true

i.e $3^{2^k} - 1$ is divisible by 2^{k+2} , but not divisibly by 2^{k+3}

$\Rightarrow 3^{2^k} - 1 = m2^{k+2}$, where m is odd so that $P(k)$ is not divisibly by 2^{k+3}

consider $P(k+1)$

$$= 3^{2^{k+1}} - 1 = (3^{2^k})^2 - 1 = (m2^{k+2} + 1)^2 - 1$$

(substitute true value of 3^{2^k} from $P(k)$)

$$= m^2 2^{2k+4} + 2m2^{k+2} = 2^{k+3} [m^2 2^{k+1} + m]$$

$$= n2^{2k+3} \quad n \text{ is odd as } m \text{ is odd}$$

$\Rightarrow P(k+1)$ is divisibly by 2^{k+3} , but not divisibly by 2^{k+4} as is odd

$\Rightarrow P(k+1)$ is true. $\Rightarrow P(x)$ is true for all n .

$$(b) (1 - \tan^4 \theta) + 2^{\tan^2 \theta} = 0$$

$$\Rightarrow 2^{\tan^2 \theta} = \tan^4 \theta - 1$$

$$\text{let } \tan^2 \theta = t \Rightarrow 2^t = t^2 - 1$$

as $t \geq 0$, graph of $y = 2^t$ and

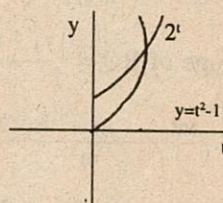
$y = t^2 - 1$ will exist in

Ist quadrant only.

From graph and by inspection

$$2^t = t^2 - 1 = 8 \Rightarrow t = 3$$

$$\Rightarrow \tan^2 \theta = 3 \Rightarrow \tan \theta \pm \sqrt{3} \Rightarrow \theta \pm \pi/3.$$



17. In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats?

Soln.:

Van can be selected in 2C_1 ways

Out of two possible arrangements of adjacent sitting, select one in 2C_1 ways. Out of remaining 11 seats, select 9 for 9 boys in ${}^{11}C_9$ ways.

So possible arrangements of sitting (3girls + 9 boys)
 $= {}^2C_1 \times 2 \times {}^{11}C_9 \times 3! \times 9! = 12!$

$$P(\text{required sitting arrangement}) = \frac{12!}{{}^{14}C_{12} 12!}$$

$$= \frac{1}{91}$$

18. A circle passes through three points A, B and C with the line segment AC as its diameter. A line passing through A intersects the chord BC at a point D inside the circle. If angles DAB and CAB and α and β respectively and the distance between the point A and the mid point of the line segment DC is d, prove that the area of the circle is

$$\frac{\pi d^2 \cos^2 \alpha}{\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta \cos(\beta - \alpha)}$$

Soln.:

In $\triangle ABC$ $AB = 2R \cos \beta$;

$BC = 2R \sin \beta$

In $\triangle ABD$

$BD = 2R \cos \beta \tan \alpha$

$CD = BC - BD$

$= 2R(\sin \beta - \cos \beta \tan \alpha)$

$$\Rightarrow MC = \frac{CD}{2} = R(\sin \beta - \cos \beta \tan \alpha)$$

In $\triangle ACM$, apply cosine rule

$$d^2 = 4R^2 + 4R^2(\sin \beta - \cos \beta \tan \alpha)^2 - 4R^2(\sin \beta - \cos \beta \tan \alpha)$$

$$\Rightarrow R^2 = \frac{d^2 \cos^2 \alpha}{4 \cos^2 \alpha + \sin^2(\beta - \alpha) - 4 \sin \beta \cos \alpha (\sin \beta - \alpha)}$$

Area = πR^2

$$= \frac{\pi d^2 \cos^2 \alpha}{4 \cos^2 \alpha + \sin^2(\beta - \alpha) - 2[\sin \alpha + \beta + \sin \beta - \alpha] \sin(\beta - \alpha)}$$

$$\text{Area} = \frac{\pi d^2 \cos^2 \alpha}{\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta \cos(\beta - \alpha)}$$

(on simplification)

19. This question contains three incomplete statements. In each part, fill in the blanks so that each of the resulting statements is correct. Write your answers in your answer book in the order in which the statements are given below:

(i) A nonzero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors

$\hat{i}, \hat{i} + \hat{j}$ and the plane determined by the vectors $\hat{i} - \hat{j}, \hat{i} + \hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is.....

(ii) For $n > 0$

$$\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \dots$$

(iii) If $x e^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} = \dots$

Soln.:

(i) $\pi/4$

(ii) π^2

(iii) 1

20. The question contains three incomplete statements. In each part, fill in the blanks so that each of the resulting statements is correct. Write your answers in your answer book in the order in which the statements are given below :

(i) If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector, then

$$(\vec{a} \cdot \vec{b})\vec{b} + (\vec{a} \cdot \vec{c})\vec{c} = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|} \vec{b} \times \vec{c} = \dots$$

(ii) If $f(x) = \sin^2 x +$

$\sin^2(x + \frac{\pi}{3}) + \cos x \cos(x + \frac{\pi}{3})$ and $g(\frac{5}{4}) = 1$, then $(g \cdot f)(x) = \dots$

(iii) In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is

Soln.:

(i) The question is wrong.

(ii) $\sin^2 x + \sin^2(x + \frac{\pi}{3}) + \cos x \cos(x + \frac{\pi}{3})$

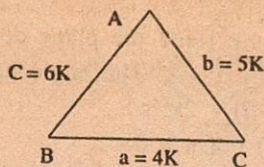
$$= \frac{1}{2} \left[1 - \cos 2x + 1 - \cos(2x + \frac{2\pi}{3}) + \cos(2x + \frac{\pi}{3}) + \cos \frac{\pi}{3} \right]$$

$$= \frac{1}{2} \left[2 - 2 \cos(2x + \frac{\pi}{3}) \times \frac{1}{2} \cos(2x + \frac{\pi}{3}) + \frac{1}{2} \right]$$

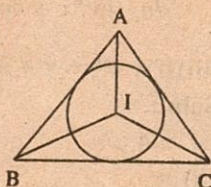
$$= \frac{1}{2} \times \frac{5}{2} = \frac{5}{4}$$

$$\therefore (g \cdot f)(x) = g[f(x)] = g(5/4) = 1$$

$$\begin{aligned} \text{(iii)} \quad \Delta &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \frac{abc}{2R} \\ \therefore R &= \frac{abc}{4\Delta} \quad \dots(1) \end{aligned}$$



$$\begin{aligned} \Delta &= \Delta IBC + \Delta ICA + \Delta IAB \\ &= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr \\ &= \frac{1}{2} (a + b + c)r = \frac{1}{2} 2sr \\ \therefore r &= \Delta/s \quad \dots(2) \end{aligned}$$



$$\begin{aligned} \frac{R}{r} &= \frac{\frac{abc}{4\Delta}}{\frac{\Delta}{s}} \times \frac{s}{\Delta} \\ &= \frac{Sabc}{4\pi r} = \frac{Sabc}{4s(s-a)(s-b)(s-c)} \\ &= \frac{4K \cdot 5K \cdot 6K}{4(15/2 K - 4K)(15K/2 - 5K)(15K/2 - 6K)} \\ &= \frac{16}{7} \end{aligned}$$

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QUANTITATIVE APTITUDE TEST

The objective of this test is to check your basic concepts and is relevant for those taking GMAT/GRE/CAT/MBA.

— Editor

TIME : 30 Min.

1. A fruit seller has a certain number of apples of which 5% are rotten. He sells 93% of the remainder and then has 266 left. How many apples he had originally?
(a) 1000 (b) 2000
(c) 3000 (d) 4000
(e) 5000.
2. By how much does $\frac{8}{34}$ exceed $\frac{8}{4}$?
(a) $8\frac{3}{4}$ (b) 1
(c) 10 (d) 12
(e) $10\frac{2}{3}$.
3. A manufacturer's list price is 40% above the cost of manufacture. He allows a trade discount of 10% of the list price. What is his percent profit based on cost price?
(a) 30% (b) 28%
(c) 25% (d) 26%
(e) 38%.
4. What are all the values of r for which $r^2 < 16$?
(a) $r < 16$ (b) $r > -4$
(c) $r > 16$ (d) $4 < r$
(e) $-4 < r < 4$.
5. In how many ways can 3 prizes be given to 7 boys when each boy is eligible for any of the prizes?
(a) 21 (b) 343
(c) 243 (d) 7
(e) 3.
6. If Ramesh knows that P is an integer greater than 2 and less than 7 and Rajendra knows that P is an integer greater than 5 and less than 10, then Ramesh and Rajendra may conclude that....
(a) P may be any of 4 values
(b) P may be any of 3 values
(c) P may be either of 2 values
(d) P can be exactly determined
(e) there is no value of P satisfying these conditions.
7. If a shirt cost Rs. 64 after a 20% discount, what was its original price (Rs.)?
(a) 76.80 (b) 80
(c) 88 (d) 86.80
(e) 51.20.
8. By selling oranges at 32 for a rupee, a man loses 40%. How many for a rupee should he sell in order to gain 20%?
(a) 10 (b) 16
(c) 13 (d) 20
(e) 25.
9. Find the value of $64^{\frac{1}{2}} + 4^{1.5} + 2^{2.5} + 27^{\frac{1}{3}}$
(a) 32.984 (b) 18.642
(c) 24.656 (d) 20.329
(e) 30.865.
10. A train running between two towns arrives at its destination 10 minutes late when it goes 40 miles/hr., and 16 minutes late when it goes 30 miles/hr. The distance between the towns is....
(a) 12 miles (b) 720 miles
(c) $8\frac{6}{7}$ miles (d) 24 miles
(e) none of these.
11. The sides of a triangle ABC together are 61 km.

- long. BC is $\frac{5}{6}$ th of AB and 3 km. longer than CA. Find the greatest side (km.)
- (a) 24 (b) 20
(c) 17 (d) 25
(e) 18.
12. If $r = 5q$, how many tenths of r does $\frac{1}{2}$ of q equal?
- (a) 4 (b) 5
(c) 1 (d) 2
(e) 3.
13. 4 litres of a certain mixture of alcohol and water is at 50% strength. It is added with one litre of water. The alcohol strength of the new mixture is....
- (a) 40% (b) 25%
(c) 20% (d) 12.5%.
14. A watch reads 4.30. If the minute hand points East, in what direction will the hour hand point?
- (a) north (b) north-west
(c) north-east (d) south-east.
15. The price of sugar having risen by 60%, by how much per cent must a householder reduce his consumption of sugar so as not to increase his expenditure?
- (a) 60% (b) 40%
(c) 20% (d) $37\frac{1}{2}\%$.
16. A cup of milk contains 3 parts pure milk and 1 part water. How much of the mixture must be withdrawn and substituted with water in order that the resulting mixture may be half milk and half water?
- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{4}$ (d) $\frac{1}{5}$.
17. A piece of wire 132 cm. long is bent successively in the shape of an equilateral triangle, a square, a regular hexagon and a circle. The area included is largest when the shape is a ...
- (a) triangle (b) square
(c) hexagon (d) circle.
18. If n is even, $(12)^n - 1$ is divisible by.....
- (a) 12 - 2 (b) 12 - 1
(c) 12 (d) $12 + n$.
19. In a quadrilateral ABCD, the sides and diagonal are related as....
- (a) $AB + BC + CD + DA < AC + BD$
(b) $AB + BC + CD + DA = AC + BD$
(c) $AB + BC + CD + DA > AC + BD$
(d) $AB + BC + CD + DA > 2(AC + BD)$.
20. In a right angled triangle, if the square of the hypotenuse is equal to twice the product of the other two sides, one of the acute angles of the triangle is....
- (a) 60° (b) 45°
(c) 30° (d) 15° .
21. A and B are the centres of two circles whose radii are 5 cm. and 2 cm. respectively. The direct common tangents of the circles meet AB in P. Then P divides AB.....
- (a) internally in the ratio 5 : 2
(b) internally in the ratio 2 : 5
(c) externally in the ratio 5 : 2
(d) externally in the ratio 2 : 5.
22. In an exam, the average marks obtained by Rajiv in English, Maths, Hindi and Drawing were 50. His average marks in Maths, Science, Social Studies and Craft were 70. If the average marks in all seven subjects were 58, his score in Maths was.....
- (a) 50 (b) 52
(c) 60 (d) 74.
23. The area of the triangle with side 4, 13, 15 units is....
- (a) 420 (b) 24
(c) 48 (d) 56.
24. Two numbers are in the ratio 5 : 6. When 4 is subtracted from each, the ratio becomes 4 : 5. The greater number is ...
- (a) 11 (b) 20
(c) 24 (d) 30.
25. The fraction of volume of a cone left after

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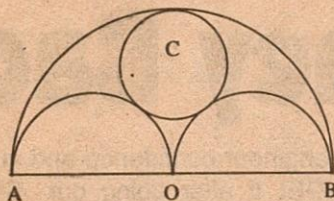


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cutting it by a plane parallel to its base at half the height is....

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{4}$ (d) $\frac{7}{8}$.

26. In the fig. O is the centre of the semicircle on AB as diameter. A circle with centre C is drawn to



touch the semicircle and also the semicircles on diameters, AO and OB. If $OA = 6$ cm., the radius of the circle with centre C is....

- (a) 2 cm. (b) 3 cm.
(c) 3.5 cm. (d) 4 cm.

27. Four children A, B, C, D divide a bag of sweets, A takes $\frac{1}{3}$ of them; B $\frac{2}{5}$ th of the remainder and the rest is equally shared between C and D. What fraction of the original did C or D get?

- (a) $\frac{1}{5}$ (b) $\frac{1}{17}$
(c) $\frac{1}{6}$ (d) $\frac{1}{4}$.

28. A boy who was asked to find $3\frac{1}{2}\%$ of a sum of money misread the question and found $5\frac{1}{2}\%$ of it. His answer was Rs. 220; what would have been the correct answer?

- (a) 120 Rs. (b) 140 Rs.
(c) 160 Rs. (d) 150 Rs.

29. $\frac{2.3^3 - 0.027}{(2.3)^2 + 0.69 + 0.09} = ?$

- (a) 0 (b) 2.0
(c) 1.6 (d) 3.4.

30. The average height in a class of 25 students is 140 cm. Five more students join the class and the average height increases to 145 cm. The average height of new entrants is....

- (a) 135 cm. (b) 150 cm.
(c) 165 cm. (d) 170 cm.

31. A certain card game has a special pack. If you

deal out equally to five players there are two cards left undealt. The same is true if you deal to seven players. If the cards are dealt to three players there is one left undealt. Number of cards that cannot be there in the pack?

- (a) 37 (b) 142
(c) 247 (d) 150.

32. Present age of Anand is 47 years and his wife's age is just 38 years. When they were married, Anand was exactly one and half times as old as his wife. How many years ago were they married?

- (a) 15 (b) 20
(c) 25 (d) 30.

33. A student rides on bicycle at 8 km/hr. and reach his school 2.5 minutes late. The next day he increases his speed to 10 km/hr and reaches school 5 minutes early. How far is the school from his house?

- (a) $5\frac{1}{4}$ km. (b) 8 km.
(c) 5 km. (d) 10 km.
(e) 7.5 km.

34. If the A.M between a and b is twice as great as the G.M., then a : b is

- (a) $2 - \sqrt{3} : 2 + \sqrt{3}$ (b) $2 + \sqrt{3} : 2 - \sqrt{3}$
(c) both in (a) and (b) (d) $\sqrt{3} + 2 : \sqrt{3} - 2$
(e) none of these.

35. The cost of setting up type of a magazine is Rs. 1000. the cost of running the printing machines is Rs. 120 per 100 copies. The cost of paper, ink etc. is 60 paise per copy. The magazines are sold at Rs. 2.75 each. 900 copies are printed, but only 784 copies are sold. What is the sum to be obtained from advertisements to give a profit of 10% on the sales?

- (a) Rs. 572 (b) Rs. 464
(c) Rs. 654 (d) Rs. 710
(e) Rs. 726.

36. From point P outside a circle, with a circumference of 10 units, a tangent is drawn. Also from P a secant is drawn dividing the circle into unequal arcs with lengths m and n. It is found that t, the length of the tangent, is the mean proportional between m and n.

If m , n and t are integers, then t may have the following number of values

- (a) zero (b) one
(c) two (d) three
(e) infinitely many.

37. In this diagram AB and AC are the equal sides of an isosceles triangle ABC , in which is inscribed an equilateral triangle DEF . Designate angle BFD by a , angle ADE by b , and Angle FEC by c . Then:

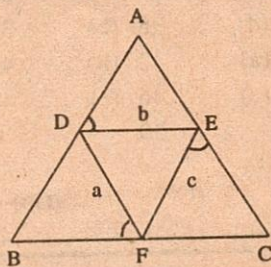
(a) $b = \frac{a+c}{2}$

(b) $b = \frac{a-c}{2}$

(c) $a = \frac{b-c}{2}$

(d) $a = \frac{b+c}{2}$

- (e) none of these.



38. On level ground, the angle of elevation of the top of the tower is 30° . On moving 20 metres nearer, the angle of elevation is 60° . The height of the tower in metres is

- (a) $20\sqrt{3}$ (b) $10\sqrt{3}$
(c) $10(\sqrt{3} + 1)$ (d) $10(\sqrt{3} - 1)$
(e) none of these.

39. From a container having pure milk, 20% is replaced by water and the process is repeated thrice. At the end of the third operation the milk is

- (a) 40% pure (b) 50% pure
(c) 51.2% pure (d) 58.8% pure
(e) 45% pure.

40. On being asked his age, Ram replied, "Take my age three years hence, multiply it by 3 and then subtract three times my age three years ago and you know how old I am". How old is Ram?

- (a) 12 years (b) 15 years
(c) 18 years (d) 24 years
(e) cannot be determined.

41. A rural belt has been divided into five zones A, B, C, D and E depending upon the soil conditions

in the region. The population (which is the same for all the 5 zones) in these zones has been classified into two major activities: cultivation and milk producing. In zone A, 80% of the population is engaged in cultivation and 50% in milk producing. The respective percentage for these activities in zone B, C, D and E are 85% and 50%, 75% and 30%, 90% and 20%, 60% and 60% respectively. What percentage of the population over all the zones is engaged in milk producing alone?

- (a) 22% (b) 40%
(c) 42% (d) 50%
(e) none of these.

42. A does a piece of work in 6 days, B in 8 days, C in 10 days. A alone worked for 2 days and then B and C together worked for 2 days and B then left. The remaining work was completed by C. How long did the work take?

- (a) 6 days (b) $6\frac{1}{6}$ days
(c) $5\frac{3}{4}$ days (d) $7\frac{1}{3}$ days
(e) none of these.

43. The coordinates of two points A and B are respectively (3, -2) and (-4, 6). The point P on the line AB divides it externally in such a way that $AP/AB = 2/5$. Find the co-ordinates of the point P.

- (a) (2/5, 3/5) (b) (1/5, 6/5)
(c) (3/5, 6/5) (d) (4/5, 7/5)
(e) (29/5, -26/5).

44. A swimming pool is 25 metres long. Its depth, d metre, at any point x metre from the edge at the shallow end is given by the formula.

$$d = 1 + \frac{3x}{25}$$

A man whose nose-height is 1.90 m. walks into the pool. Find the farthest distance, from the edge at the shallow end, he can go without getting his nose wet. Assume that he keeps his head vertical all the time.

- (a) 6.9 (b) 7.5 m
(c) 7.3 m (d) 6.6 m.
(e) none of these.

45. A solid cube of side 1 cm. is heated such that

its edges increase by 50%, then its surface area will increase by

- (a) 75% (b) 37.5%
(c) 125% (d) 112.5%
(e) 50%.

46. The sum of first n terms of an A.P., whose first term is π is zero. The sum of next m terms is

- (a) $\frac{\pi m(m+n)}{n-1}$ (b) $\frac{\pi m(m+n)}{1-n}$
(c) $\frac{\pi n(m+n)}{1-n}$ (d) $\frac{\pi(m+n)n}{n-1}$
(e) none of these.

47. The sum of the series $0.4 + 0.44 + 0.444 + 0.4444 + \dots$ upto n terms, is

- (a) $\frac{4}{81} \left(9n + 1 + \frac{1}{10^n} \right)$ (b) $\frac{4}{9} \left(9n - 1 + \frac{1}{10^n} \right)$
(c) $\frac{4}{81} \left(9n - 1 + \frac{1}{10^n} \right)$ (d) $\frac{4}{90} \left(9n - 1 - \frac{1}{10^n} \right)$
(e) none of these.

48. The roots of the equation $x^2 - 2\sqrt{3}x + 3 = 0$ are

- (a) rational and unequal
(b) irrational and unequal
(c) rational and equal (d) real and equal
(e) none of these.

49. A factory employs skilled workers, unskilled workers and clerks in the proportion $8 : 5 : 1$ and the wages of a skilled worker, an unskilled worker and a clerk are in the ratio $5 : 2 : 3$. When 20 unskilled workers are employed, the total daily wages of all, amount to Rs. 318. Find the daily wages paid to each category of employees (Rs.)

- (a) 240, 56, 18
(b) 230, 65, 12
(c) 210, 70, 15
(d) 240, 60, 18
(e) none of these.

50. Find the value of $(27)^{-2/3} + [(2^{-2/3})^{-5/3}]^{-9/10}$

- (a) $11/18$ (b) $7/18$
(c) $6/11$ (d) $8/13$
(e) none of these.

Answers

- | | | | |
|---------|---------|---------|---------|
| 1. (d) | 2. (c) | 3. (d) | 4. (e) |
| 5. (b) | 6. (d) | 7. (b) | 8. (b) |
| 9. (c) | 10. (a) | 11. (a) | 12. (c) |
| 13. (a) | 14. (c) | 15. (d) | 16. (a) |
| 17. (d) | 18. (b) | 19. (c) | 20. (b) |
| 21. (c) | 22. (d) | 23. (b) | 24. (c) |
| 25. (d) | 26. (a) | 27. (a) | 28. (b) |
| 29. (b) | 30. (d) | 31. (d) | 32. (b) |
| 33. (c) | 34. (e) | 35. (e) | 36. (c) |
| 37. (d) | 38. (b) | 39. (c) | 40. (c) |
| 41. (a) | 42. (b) | 43. (e) | 44. (b) |
| 45. (c) | 46. (e) | 47. | 48. |
| 49. | 50. | | |



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1. If α and β are the roots of $ax^2 + bx + c = 0$, where a, b, c are real, prove that $\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n$ is always real for every integer n .

2. Show that $1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = 0$ cannot have repeated roots.

3. Sum to n terms

$$\sin^3 \theta + \frac{\sin^3 3\theta}{3} + \frac{\sin^3 3^2 \theta}{3^2} + \dots$$

4. If $Z_n = \cos \frac{n\pi}{2^n} + i \sin \frac{n\pi}{2^n}$, find $\lim_{n \rightarrow \infty} (Z_1 Z_2 \dots Z_n)$

5. Let $a_1, a_2, \dots, a_n$ and n be distinct real numbers such that

$$\left(\sum_{r=1}^{n-1} ar^2\right)x^2 + 2\left(\sum_{r=1}^{n-1} ar \cdot a_{r+1}\right)x + \sum_{r=2}^n a_r^2 \leq 0, \text{ show that } a_1, a_2, \dots, a_n \text{ are in G.P.}$$

6. Solve $\log_{\log_2(0.5x)}(x^2 - 10x + 22) > 0$

7. How many five digit numbers are there in whose notation each successive digit exceeds its predecessor?

8. Find the coefficient of x^5 in the expansion of $[(1+x)^2 + 2x^3]^5$.

9. If $\tan \alpha = 2$, find all the solutions of the equation $\frac{\tan(2\alpha - x)}{\tan(2\alpha + x)} = 11$ that lie between 0 and 2π .

10. Show that $\sec \frac{2\pi}{7}, \sec \frac{4\pi}{7}, \sec \frac{8\pi}{7}$ are the roots of $x^3 + 4x^2 - 4x - 8 = 0$.

11. Find the sum of infinite terms of the series

$$\cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \dots$$

12. If $r = R/2$, prove that the triangle is equiangular.

13. Express

$$(\beta^3/2) \operatorname{cosec}^2[(1/2)\tan^{-1}(\beta/\alpha)] + (\alpha^3/2) \sec^2[(1/2)\tan^{-1}(\alpha/\beta)]$$

as an integral polynomial in α and β .

14. Given ' n ' straight lines and a fixed point O . Through O is drawn a straight line meeting these n lines in $R_1, R_2, \dots, R_n$; and on the line a point R is taken such that OR is the harmonic mean of the n quantities $OR_1, OR_2, \dots, OR_n$. Show that the locus of R is a straight line.

15. In any circle prove that the perpendicular from any point on it on the line joining the points of contact of two tangents is a mean proportional between the perpendiculars from that point upon the two tangents.

16. Find the equation of the chord of the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ which is trisected at the points $(1/3, 1/3)$ and $(8/3, 8/3)$.

17. Find the equation of the straight line passing through the intersection of the lines $x + y - 5 = 0$ and $2x - y + 4 = 0$ and meeting the axes in the points A and B such that the triangle OAB is isosceles.

18. Evaluate : $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{(4 + \cos \theta)^{\frac{1}{2}} - 2}{\theta - \frac{\pi}{2}}$

19. Evaluate : $\lim_{m \rightarrow 0} \frac{\sqrt[3]{8+m^2+m} - \sqrt[3]{8+m}}{\sqrt[3]{8+m} - \sqrt[3]{8+m^2-m^3}}$

20. Find

$$\frac{d}{dx} \left(\int_{\sqrt{x}}^{x^2+x} \frac{dt}{2+\sqrt{t}} \right)$$

21. Find $\frac{d^2y}{dx^2}$ given that $y^3 - x^2 = 4$.

22. Find the intervals on which $f(x) = |x+1| |x-2|$ increases and the intervals on which it decreases.

23. Expressing the limit of a sum in the form of a

definite integrals evaluate $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n^3}{(n^2+r^2)(n^2+2r^2)}$

24. Solve the differential equation

$$(1 + 3e^{y/x})dy + 3e^{y/x} \left(1 - \frac{y}{x}\right)dx = 0$$

25. Find $\int \sqrt{1 - e^x} dx$

26. Find the largest rectangle which can be inscribed in the area bounded by $y = 0$ and the semi-circle $y = \sqrt{1 - x^2}$.

27. Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \text{ the circle } x^2 + y^2 = a^2, \text{ and the lines } x = |c|, c \leq a.$$

28. Evaluate : $\int \frac{1 + \sin 2x}{\sin x(1 + \cos x)} dx$

29. Evaluate : $\int \frac{dx}{(x^2 + 9)^2}$

30. Examine the continuity and differentiability of: $f(x) = |x| + |x-1| + |x-2|$. ($x \in \mathbb{R}$).

31. Prove that $\sqrt{3}$ is not a rational number.

32. Let $S_n = \frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \dots + \frac{1+2+\dots+n}{1^3+2^3+\dots+n^3}$,

$n = 1, 2, 3, \dots$ then S_n is not greater than

(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) 4.

33. The real roots of $|x|^3 - 3x^2 + 3|x| - 2 = 0$ are

(a) 0, 2

(c) ± 2

(b) ± 1

(d) 1, 2.

34. The simultaneous equations

$$kx + 2y - z = 1$$

$$(k-1)y - 2z = 2$$

$$(k+2)z = 3 \text{ have only one solution when}$$

(a) $k = -2$

(b) $k = -1$

(c) $k = 0$

(d) $k = 1$.

35. The product of all values of $(\cos \alpha + i \sin \alpha)^{3/5}$ is

(a) 1

(b) $\cos \alpha + i \sin \alpha$

(c) $\cos 3\alpha + i \sin 3\alpha$

(d) $\cos 5\alpha + i \sin 5\alpha$.

36. The equation $\cos p\theta + \cos(p+2)\theta = \cos \theta$ is satisfied by

(a) $\theta = 0$

(b) $\theta = \pi$

(c) $\theta = -\pi$

(d) $\theta = -\pi/2$.

37. The area bounded by

$$y = x^2, y = [x+1], x \leq 1 \text{ and the } y\text{-axis is}$$

(a) $1/3$

(b) $2/3$

(c) 1

(d) $7/3$.

38. State true or false showing necessary calculations: The domain of definition of

$$f(x) = \sqrt{\frac{3x^2 + 18x + 29}{x+3}} - 2^{6x+17} \text{ is only all positive real } x.$$

39. Find the value of

$$\sin^{-1} \left\{ \cot \left(\sin^{-1} \sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1} \frac{\sqrt{12}}{4} + \sec^{-1} \sqrt{2} \right) \right\}$$

40. Find the no. of ways in which 7 pictures can be hung from 5 picture nails on the wall.

SOLUTIONS

1. For $n = 0$, $\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n = 2$, which is real. For n as a positive integer we prove by induction. For $n = 1$,

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha + \beta)^2}{\alpha\beta} - 2 = \frac{b^2}{ac} - 2$$

which is real. We assume that the result is true for all +ve integers upto $n = k$. Now, we shall show that it is true for $n = k + 1$ also. Indeed

$$\left(\frac{\alpha}{\beta}\right)^{k+1} + \left(\frac{\beta}{\alpha}\right)^{k+1} = \frac{\alpha^{2k+2} + \beta^{2k+2}}{(\alpha\beta)^{k+1}}$$

$$= \underbrace{\left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)}_{\text{Real}} \underbrace{\left(\frac{\alpha^{2k} + \beta^{2k}}{(\alpha\beta)^k}\right)}_{\text{Real}} - \underbrace{\frac{\alpha^{2k-2} + \beta^{2k-2}}{(\alpha\beta)^{k-1}}}_{\text{Real}}$$

which is real.

Now we shall show that the result holds when n is a -ve integer. Put $n = -m$, where m is a +ve inter. Then,

$$\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n = \left(\frac{\alpha}{\beta}\right)^m + \left(\frac{\beta}{\alpha}\right)^m,$$

which is real as shown above.

2. Let $f(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

$f(0) \neq 0$, Therefore 0 is not a root of $f(x) = 0$. Let us assume that α is a repeated root of $f(x) = 0$. Then α must be a root of $f'(x) = 0$ also

$$f(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \dots + \frac{\alpha^n}{n!} = 0 \quad \dots(1)$$

$$\text{and } f'(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \dots + \frac{\alpha^{n-1}}{(n-1)!} = 0 \quad \dots(2)$$

Subtracting (2) from (1), we have

$\frac{\alpha^n}{n!} = 0 \Rightarrow \alpha = 0$, implying that 0 is a repeated root of $f(x) = 0$, a contradiction since 0 is not a root of $f(x) = 0$, at all. Hence $f(x) = 0$ does not have repeated roots.

3. $(r+1)$ th term of the given series = $\frac{\sin^3 3^r \theta}{3^r}$

$$= \frac{1}{3^r} \left(\frac{3 \sin 3^r \theta - \sin 3^{r+1} \theta}{4} \right)$$

$$\left(\because \sin^3 \theta = \frac{3 \sin \theta - \sin^3 \theta}{4} \right)$$

$$= \frac{1}{4} \cdot \frac{1}{3^{r-1}} \cdot \sin 3^r \theta - \frac{1}{4} \cdot \frac{1}{3^r} \sin 3^{r+1} \theta$$

1st term = $\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$

2nd term = $\frac{1}{4} \sin 3\theta - \frac{1}{4} \cdot \frac{1}{3} \sin 3^2 \theta$

3rd term = $\frac{1}{4} \cdot \frac{1}{3} \sin 3^2 \theta - \frac{1}{4} \cdot \frac{1}{3^2} \sin 3^3 \theta$

n th term = $\frac{1}{4} \cdot \frac{1}{3^{n-2}} \sin 3^{n-1} \theta - \frac{1}{4} \cdot \frac{1}{3^{n-1}} \sin 3^n \theta$

Adding vertically

sum to n terms = $\frac{3}{4} \sin \theta - \frac{1}{4} \cdot \frac{1}{3^{n-1}} \sin 3^n \theta$

4. $Z_1 Z_2 \dots Z_n = \cos \pi S_n + i \sin \pi S_n$

where $S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n}$

$\lim_{n \rightarrow \infty} (Z_1 Z_2 \dots Z_n) = \cos \pi \left(\lim_{n \rightarrow \infty} S_n \right) + i \sin \pi \left(\lim_{n \rightarrow \infty} S_n \right)$

where $S = \lim_{n \rightarrow \infty} S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} + \dots$

$$\Rightarrow \frac{1}{2} S = \frac{1}{2^2} + \frac{2}{2^3} + \dots + \frac{n-1}{2^n} + \dots$$

$$\Rightarrow (1 - \frac{1}{2})S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \Rightarrow S = 2$$

Hence $\lim_{n \rightarrow \infty} (Z_1 \cdot Z_2 \cdot Z_3 \dots Z_n) = \cos 2\pi + i \sin 2\pi = 1$

5. $(a_1^2 + a_2^2 + \dots + a_{n-1}^2)x^2 + 2(a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n)x + (a_2^2 + a_3^2 + \dots + a_n^2) \leq 0$

$$\Rightarrow (a_1 x + a_2)^2 + (a_2 x + a_3)^2 + \dots + (a_{n-1} x + a_n)^2 \leq 0 \quad \dots(1)$$

Also, since $a_1, a_2, \dots, a_n$ and x are distinct real numbers,

$$(a_1 x + a_2)^2 + (a_2 x + a_3)^2 + \dots + (a_{n-1} x + a_n)^2 \geq 0 \quad \dots(2)$$

(1) and (2) can be simultaneously true when

$$(a_1 x + a_2)^2 + (a_2 x + a_3)^2 + \dots + (a_{n-1} x + a_n)^2 = 0$$

$$\Rightarrow -x = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_n}{a_{n-1}}$$

$\Rightarrow a_1, a_2, a_3, \dots, a_n$ are in G.P.

6. We need to find solution of

$$\log_{\log_2(0.5x)}(x^2 - 10x + 22) > 0$$

if $\log_2(0.5x) > 1$, i.e. if $0.5x > 2$

i.e. if $x > 4$ the inequality given is

$$x^2 - 10x + 22 > 1$$

$$\Rightarrow x^2 - 10x + 21 > 0$$

This gives values outside 3 and 7 of the roots. This, with the earlier condition of $x > 4$, gives values of x in $(7, +\infty)$

Next, if $\log_2(0.5x) < 1$, i.e. $x < 4$

$$x^2 - 10x + 22 < 1$$

$$\text{i.e. } x^2 - 10x + 21 < 0$$

The roots of $x^2 - 10x + 21 = 0$ are 3, 7

$\therefore$ I is satisfied if $3 < x < 7$ already $x < 4$

$\therefore$ The possible values are $3 < x < 4$.

Again $x^2 - 10x + 22 > 0$, for the logarithm to be real.

$$\text{The roots of } x^2 - 10x + 22 = 0 \text{ are } \frac{10 \pm \sqrt{100 - 88}}{2}$$

$$\text{i.e. } 5 + \sqrt{3}, 5 - \sqrt{3}.$$

The values of x are those that lie outside $5 + \sqrt{3}$ and $5 - \sqrt{3}$.

$\therefore$ This, combined with the condition $3 < x < 4$ gives values of x in $(3, 5 - \sqrt{3})$. Hence the values of x satisfying the inequality are those such that $x \in (3, 5 - \sqrt{3}) \cup (7, +\infty)$.

7. Evidently zero cannot be used in such a number, since it cannot be in the first place nor can it occupy any other subsequent place since it would contradict the condition of every successive digit exceeding its predecessor.

Hence from the remaining choose any five (This can be done in ${}^9C_5 (= {}^9C_4)$ ways) and arrange any one choice say $a_1 a_2 a_3 a_4 a_5$ in only order to form the number a_1, a_2, a_3, a_4, a_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$. There is only one such number for every choice of 5.

$\therefore$ The number of five digits is ${}^9C_4 = 126$.

$$8. [(1+x)^2 + 2x^3]^5 = (1+2x+x^2+2x^3)^5 = (1+2x)^5(1+x^2)^5$$

$$(1+2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

$$(1+x^2)^5 = 1 + 5x^2 + 10x^4 + 10x^6 + 5x^8 + x^{10}$$

Hence the coefficient of x^5 in the expansion of $(1+2x)^5(1+x^2)^5$ is $32 + 400 + 100 = 532$.

$$9. \tan \alpha = 2 \Rightarrow \tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha} = -\frac{4}{3}$$

$$\frac{\tan(2\alpha - x)}{\tan(2\alpha + x)} = 11$$

$$\Rightarrow \frac{\tan x + \frac{4}{3}}{1 - \frac{4}{3}\tan x} + 11 \frac{\tan x - \frac{4}{3}}{1 + \frac{4}{3}\tan x} = 0$$

$$\Rightarrow 2\tan^2 x - 5\tan x + 2 = 0$$

$$\Rightarrow (\tan x - \frac{1}{2})(\tan x - 2) = 0$$

$$\Rightarrow \tan x = 2 = \tan \alpha \text{ or } \tan x = \frac{1}{2}$$

$$= \cot \alpha = \tan(\pi/2 - \alpha)$$

$$\tan x = \tan \alpha$$

given

$$x = \alpha, \pi + \alpha$$

$$\tan x = \tan(\pi/2 - \alpha)$$

given

$$x = \pi/2 - \alpha, 3\pi/2 - \alpha$$

$$0 < \alpha < \pi/2$$

10. Let $\theta = 0, 2\pi/7, 4\pi/7$ or $8\pi/7$

$$\Rightarrow 7\theta = 0, 2\pi, 4\pi, \text{ or } 8\pi$$

$$\Rightarrow 4\theta = 2n\pi - 3\theta, n = 0, 1, 2, \text{ or } 4$$

$$\Rightarrow \cos 4\theta = \cos 3\theta$$

$$\Rightarrow 2(2\cos^2 \theta - 1)^2 = 4\cos^3 \theta - 3\cos \theta$$

$$\Rightarrow 8\cos^4 \theta - 4\cos^3 \theta - 8\cos^2 \theta + 3\cos \theta + 1 = 0$$

$$\Rightarrow \sec^4 \theta + 3\sec^3 \theta - 8\sec^2 \theta - 4\sec \theta + 8 = 0$$

Thus $\sec \theta (= 1)$, $\sec 2\pi/7$, $\sec 4\pi/7$

$\sec 8\pi/7$ are the roots of

$$x^4 + 3x^3 - 8x^2 - 4x + 8 = 0$$

$$\text{or } (x-1)(x^3 + 4x^2 - 4x - 8) = 0$$

Hence $\sec(2\pi/7)$, $\sec(4\pi/7)$, $\sec(8\pi/7)$ are the roots of $x^3 + 4x^2 - 4x - 8 = 0$.

11. The formation of 2, 8, 18, 32, indicates that the n th term in the sequence is $2n^2$.

The n th term of the given series

$$= \cot^{-1}(2n^2) = \tan^{-1}\left(\frac{1}{2n^2}\right)$$

$$= \tan^{-1} \left\{ \frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} \right\}$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$$\therefore \text{1st term} = \tan^{-1}3 - \tan^{-1}1$$

$$\text{2nd term} = \tan^{-1}5 - \tan^{-1}3$$

.....

$$\therefore \text{nth term} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$$\therefore \text{sum to } n \text{ terms} = \tan^{-1}(2n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}(2n+1) - \tan \pi/4$$

$$\text{As } n \rightarrow \infty, (2n+1) \rightarrow \infty$$

$$\therefore \tan^{-1}(2n+1) \rightarrow \pi/2$$

$$\therefore \text{sum to infinite terms}$$

$$= \pi/2 - \pi/4 = \pi/4$$

$$12. R = 2r \text{ But } r = 4R \cdot \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}$$

$$\therefore R = 2 \cdot 4R \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}$$

$$\text{or, } 8 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = 1$$

$$\text{or, } 4 \left(2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \right) \cdot \sin \frac{C}{2} = 1$$

$$\text{or, } 4 \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] \cdot \sin \frac{C}{2} = 1$$

$$\text{or, } 4 \cos \frac{A-B}{2} \cdot \sin \left(\frac{\pi}{2} - \frac{A+B}{2} \right) - 4 \sin \left(\frac{\pi}{2} - \frac{A+B}{2} \right) \cdot \sin \frac{C}{2} = 1$$

$$\text{or, } 4 \cos \frac{A+B}{2} \cdot \cos \frac{A+B}{2} - 4 \sin^2 \frac{C}{2} = 1$$

$$\text{or, } 2 \cos A + 2 \cos B - 2(1 - \cos C) = 1$$

$$\text{or, } 2 \cos A + 2 \cos B + 2 \cos C = 3$$

$$\text{or, } \cos A + \cos B + \cos C = \frac{3}{2}$$

$$\text{or, } \frac{b^2 + c^2 - a^2}{2bc} + \dots + \dots = \frac{3}{2}$$

$$\text{or, } \frac{a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c(a^2 + b^2 - c^2)}{2abc} = \frac{3}{2}$$

$$\text{or, } ab^2 + c^2a - a^3 + bc^2 + a^2b - b^3 + ca^2 + b^2c - c^3 = 3abc$$

$$\text{or } a^3 + b^3 + c^3 - 3abc - a(b^2 + c^2) + 2abc - b(c^2 + a^2) + 2abc - c(a^2 + b^2) + 2abc = 0$$

$$\text{or } (b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) = 0.$$

In the $\triangle ABC$, $b+c > a$, $c+a > b$, $a+b > c$.

So, $(b+c-a)$, $(c+a-b)$, $(a+b-c)$ are all +ve and also $(b-c)^2(b+c-a)$, ... etc. are all +ve and since their sum is equal to zero,

$$b-c=0, a-b=0$$

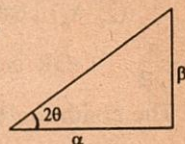
$$\text{i.e. } a=b=c$$

$$\therefore \angle A = \angle B = \angle C = 60^\circ$$

$\therefore$ The triangle is equiangular.

$$13. \text{ Let } \left(\frac{1}{2}\right) \tan^{-1}(\beta/\alpha) = 0$$

$$\text{so that } \tan 2\theta = \beta/\alpha$$



$$\text{Now, } \operatorname{cosec}^2[(1/2)\tan^{-1}(\beta/\alpha)] = \operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta}$$

$$= \frac{2}{1 - \cos 2\theta}$$

$$= \frac{2}{1 - \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \right)} = \frac{2\sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2} - \alpha}$$

$$\therefore \left(\frac{\beta^3}{2}\right) \operatorname{cosec}^2 \left[\left(\frac{1}{2}\right) \tan^{-1} \left(\frac{\beta}{\alpha} \right) \right] = \frac{\beta^3}{2} \cdot \frac{2\sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2} - \alpha}$$

$$= \beta \sqrt{\alpha^2 + \beta^2} (\sqrt{\alpha^2 + \beta^2} + \alpha) \quad \dots(1)$$

similarly

$$\therefore \left(\frac{\alpha^3}{2}\right) \sec^2 \left[\left(\frac{1}{2}\right) \tan^{-1} \left(\frac{\alpha}{\beta} \right) \right]$$

$$= \alpha \sqrt{\alpha^2 + \beta^2} (\sqrt{\alpha^2 + \beta^2} - \beta) \quad \dots(2)$$

$\therefore$ Given expression

$$= \beta \sqrt{\alpha^2 + \beta^2} (\sqrt{\alpha^2 + \beta^2} + \alpha) + \alpha \sqrt{\alpha^2 + \beta^2} (\sqrt{\alpha^2 + \beta^2} - \beta)$$

$$= (\alpha^2 + \beta^2)(\beta + \alpha) = \alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3, \text{ a polynomial in } \alpha \text{ and } \beta \text{ with integer coefficients.}$$

14. Let the fixed point O be the origin and let the n straight lines be

$$A_k x + B_k y + 1 = 0 \quad (k = 1, 2, \dots, n)$$

Let the line through O be taken as

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = r$$

For the kth line, the distance along this line of that point R_k is OR_k , and R_k is $(OR_k \cos \theta, OR_k \sin \theta)$ and this lies on $A_k x + B_k y + 1 = 0$.

$$\therefore A_k(OR_k \cos \theta) + B_k(OR_k \sin \theta) + 1 = 0$$

$$\therefore \frac{1}{OR_k} = -(A_k \cos \theta + B_k \sin \theta)$$

This result is true for all $k = 1, 2, \dots, n$

$$\therefore \frac{1}{OR_1} + \frac{1}{OR_2} + \dots + \frac{1}{OR_n} = -(\sum A_k) \cos \theta - (\sum B_k) \sin \theta$$

$$= \frac{n}{OR} \quad (\because OR \text{ is H.M of } OR_1, OR_2, \dots, OR_n)$$

$\therefore$ The condition becomes

$$\left(\frac{\sum A_k}{n}\right)OR \cos \theta + \left(\frac{\sum B_k}{n}\right)OR \sin \theta + 1 = 0$$

But R has co-ordinates $(OR \cos \theta, OR \sin \theta)$

$\therefore$ The locus of R is therefore

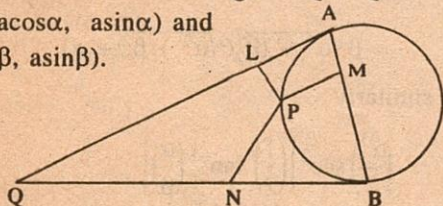
$$\left(\frac{\sum A_k}{n}\right)x + \left(\frac{\sum B_k}{n}\right)y + 1 = 0$$

which is a straight line.

15. Let the circle be $x^2 + y^2 = a^2$. $P(a \cos \theta, a \sin \theta)$ be any point on the circle. AB is the line joining the points of contact of the tangents QA, QB.

Let A be $(a \cos \alpha, a \sin \alpha)$ and

B be $(a \cos \beta, a \sin \beta)$.



PL, PM, PN are respectively perpendiculars to QA, AB and BQ.

Tangent at A is $x \cos \alpha + y \sin \alpha - a = 0$

$$PL = a - a(\cos \theta \cos \alpha + \sin \theta \sin \alpha)$$

$$PL = a(1 - \cos(\theta - \alpha)) = 2a \sin^2\left(\frac{\theta - \alpha}{2}\right)$$

$$\text{Similarly, } PN = 2a \sin^2\left(\frac{\theta - \beta}{2}\right)$$

Equation to the chord AB is

$$\frac{y - a \sin \alpha}{x - a \cos \alpha} = \frac{a(\sin \alpha - \sin \beta)}{a(\cos \alpha - \cos \beta)}$$

$$= \frac{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{-2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}} = \frac{-\cos \frac{\alpha + \beta}{2}}{\sin \frac{\alpha + \beta}{2}} \quad (\because \alpha \neq \beta)$$

This simplified, is of the form

$$x \cos\left(\frac{\alpha + \beta}{2}\right) + y \sin\left(\frac{\alpha + \beta}{2}\right) = a \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\therefore PM = a \cos\left(\frac{\alpha - \beta}{2}\right) - a \left[\cos \theta \cos \frac{\alpha + \beta}{2} + \sin \theta \sin \frac{\alpha + \beta}{2} \right]$$

$$= a \cos\left(\frac{\alpha - \beta}{2}\right) - a \left[\cos\left(\theta - \frac{\alpha + \beta}{2}\right) \right]$$

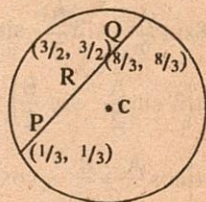
$$= a \left[\cos \frac{\alpha - \beta}{2} - \cos\left(\theta - \frac{\alpha + \beta}{2}\right) \right]$$

$$= 2a \sin \frac{\theta - \beta}{2} \sin \frac{\theta - \alpha}{2}$$

$$PL \cdot PN = 4a^2 \sin^2\left(\frac{\theta - \alpha}{2}\right) \sin^2\left(\frac{\theta - \beta}{2}\right)$$

$$PL \cdot PN = PM^2$$

16. Chord which is trisected at the points $P(1/3, 1/3)$ and $Q(8/3, 8/3)$ is bisected at the point $R(3/2, 3/2)$ which is also the midpoint of PQ.



Now we have to find the equation of the chord which passes through the point $R(3/2, 3/2)$ and is perpendicular to CR, where $C(2, 1)$ is the centre of the circle.

$$\text{Slope of CR} = \frac{\frac{3}{2} - 1}{\frac{3}{2} - 2} = -1$$

Slope of the chord = 1

$$\therefore \text{Equation of the chord is } y - \frac{3}{2} = 1\left(x - \frac{3}{2}\right)$$

on $x = y$.

17. Equation of any line passing through the intersection of $x + y - 5 = 0$, $2x - y + 4 = 0$ is $x + y - 5 + k(2x - y + 4) = 0$.

$$\Rightarrow \frac{x}{(5-4k)} + \frac{y}{(5-4k)} = 1$$

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Since $\triangle OAB$ is isosceles, $OA = OB$.

$$\text{Therefore } \frac{5-4k}{1+2k} = \pm \frac{5-4k}{1-k}$$

$$\therefore k = 0, -2.$$

Hence the lines are

$$x + y - 5 = 0 \text{ and } 3x - 3y + 13 = 0.$$

$$18. \text{ Put } \theta - \frac{\pi}{2} = \phi$$

$$\text{Then, } \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{(4 + \cos \theta)^{\frac{1}{2}} - 2}{\left(\theta - \frac{\pi}{2}\right)}$$

$$= \lim_{\phi \rightarrow 0} \frac{(4 - \sin \phi)^{\frac{1}{2}} - 2}{\phi}$$

$$= \lim_{\phi \rightarrow 0} \frac{[(4 - \sin \phi)^{\frac{1}{2}} - 2][(4 - \sin \phi)^{\frac{1}{2}} + 2]}{\phi[(4 - \sin \phi)^{\frac{1}{2}} + 2]}$$

$$= \lim_{\phi \rightarrow 0} \left(\frac{\sin \phi}{\phi} \right) \frac{1}{(4 - \sin \phi)^{\frac{1}{2}} + 2}$$

$$= -(+1) \cdot \frac{1}{4} = -\frac{1}{4}.$$

19. Using the formula

$$x - y = \frac{x^3 - y^3}{x^2 + xy + y^2}; \quad x \neq y,$$

we have

$$\lim_{m \rightarrow 0} \frac{\sqrt[3]{8+m^2+m^3} - \sqrt[3]{8+m}}{\sqrt[3]{8+m} - \sqrt[3]{8+m^2-m^3}}$$

$$= \lim_{m \rightarrow 0} \left[\frac{(8+m^2+m^3) - (8+m)}{(8+m) - (8+m^2-m^3)} \right]$$

$$\left[\frac{\sqrt[3]{(8+m)^2 + \sqrt[3]{(8+m)(8+m^2-m^3)} + \sqrt[3]{(8+m^2-m^3)}}}{\sqrt[3]{(8+m^2+m^3)^2 + \sqrt[3]{(8+m)(8+m^2+m^3)} + \sqrt[3]{(8+m)^2}} \right]^2$$

$$= \left[\lim_{m \rightarrow 0} \frac{m(m^2+m-1)}{m(m^2-m+1)} \right] = -1$$

$$20. \text{ Put } u = \sqrt{x}, v = x^2 + x$$

$$\text{They } \frac{d}{dx} \left(\int_u^y \frac{dt}{2 + \sqrt{t}} \right)$$

$$= \frac{d}{dx} \left(\int_0^y \frac{dt}{2 + \sqrt{t}} - \int_0^u \frac{dt}{2 + \sqrt{t}} \right)$$

$$= \frac{d}{dx} \left(\int_0^y \frac{dt}{2 + \sqrt{t}} \right) \frac{dy}{dx} - \frac{d}{dx} \left(\int_0^u \frac{dt}{2 + \sqrt{t}} \right) \frac{du}{dx}$$

$$= \frac{1}{2 + \sqrt{v}} (2x + 1) - \frac{1}{2 + \sqrt{u}} \left(\frac{1}{2\sqrt{x}} \right)$$

$$= \frac{2x + 1}{2 + \sqrt{x^2 + x}} - \frac{1}{2\sqrt{x}(2 + \sqrt{x})}$$

$$21. y^3 - x^3 = 4$$

Differentiating both sides w.r.t. x , we have

$$3y^2 \frac{dy}{dx} - 2x = 0 \quad \dots(1)$$

Differentiating again, we have

$$3y^2 \frac{d^2y}{dx^2} + 6y \left(\frac{dy}{dx} \right)^2 - 2 = 0 \quad \dots(2)$$

$$(1) \text{ gives } \frac{dy}{dx} = \frac{2x}{3y^2}$$

substituting in (2), we have

$$3y^2 \frac{d^2y}{dx^2} + 6y \left(\frac{2x}{3y^2} \right)^2 - 2 = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{6y^3 - 8x^2}{9y^5}$$

$$22. f(x) = \begin{cases} x^2 - x - 2, & x < -1 \\ 2 + x - x^2, & -1 \leq x \leq 2 \\ x^2 - x - 2, & x > 2 \end{cases}$$

$$f'(x) = \begin{cases} 2x - 1, & x < -1 \\ 1 - 2x, & -1 < x < 2 \\ 2x - 1, & x > 2 \end{cases}$$

Interval	Sign of $f'(x)$	$f(x)$ increasing or decreasing
$(-\infty, -1)$	-ve	decreasing
$(-1, 2)$	+ve	increasing
$(2, \infty)$	-ve	decreasing
	+ve	increasing.

$$23. \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{x^3}{(n^2 + r^2)(n^2 + 2r^2)}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\frac{n^3}{n^4}}{\left(\frac{n^2+r^2}{n^2}\right) \left(\frac{n^2+2r^2}{n^2}\right)}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\frac{1}{n}}{\left(1+\frac{r^2}{n^2}\right) \left(1+2\frac{r^2}{n^2}\right)}$$

Let $nh = 1$

As $n \rightarrow \infty, h \rightarrow 0$

$$= \lim_{h \rightarrow 0} \sum_{r=1}^n \frac{h}{(1+r^2h^2)(1+2r^2h^2)}$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+2x^2)} = \int_0^1 \left(\frac{2}{1+2x^2} - \frac{1}{1+x^2} \right) dx$$

$$= 2 \cdot \frac{1}{2} \int_0^1 \frac{dx}{x^2 + \frac{1}{2}} - \int_0^1 \frac{dx}{x^2 + 1}$$

$$= \sqrt{2} \left[\tan^{-1} \sqrt{2}x \right]_0^1 - \tan^{-1} 1 = \sqrt{2} \tan^{-1} \sqrt{2} - \frac{\pi}{4}$$

24. Put, $y = vx, \frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore$ From the given equations

$$(1 + 3e^v)dy + 3e^v(1 - v)dx = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{3e^v(1-v)}{(1+3e^v)}$$

$$\text{or, } v + x \frac{dv}{dx} = -\frac{3e^v(1-v)}{1+3e^v}$$

$$\text{or, } x \frac{dv}{dx} = -\left[\frac{3e^v(1-v)}{1+3e^v} + v \right]$$

$$\text{or, } x \frac{dv}{dx} = -\left[\frac{3e^v + v}{1+3e^v} \right] \quad \text{or, } \int \frac{1+3e^v}{v+3e^v} dv = \int -\frac{dx}{x}$$

$$\text{or, } \log|v + 3e^v| = -\log|x| + \log C$$

$$\text{or } |x(3e^v + v)| = C$$

$$\text{or, } x(3e^{y/x} + y/x) = C$$

$$\text{or, } 3xe^{y/x} + y = C, \text{ where } C \text{ is a constant.}$$

25. Put $\sqrt{1-e^x} = u \Rightarrow \frac{-e^x dx}{2\sqrt{1-e^x}} = du$

$$\Rightarrow 2dx = \frac{2u du}{u^2 - 1}$$

$$I = \int \sqrt{1-e^x} dx = \int \left[2 + \frac{1}{u-1} - \frac{1}{u+1} \right] du$$

$$= 2\sqrt{1-e^x} + \ln \left| \frac{\sqrt{1-e^x} - 1}{\sqrt{1-e^x} + 1} \right| + C$$

26. Let the base of an inscribed rectangle be $2x$.

Its height is then $\sqrt{1-x^2}$,

which makes the area

$$A(x) = 2x\sqrt{1-x^2}.$$

$$\Rightarrow \frac{dA}{dx} = 2\sqrt{1-x^2} - \frac{2x^2}{\sqrt{1-x^2}}$$

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$$

When x is slightly less than $\frac{1}{\sqrt{2}}$, $\frac{dA}{dx}$ is +ve

When x is slightly greater than $\frac{1}{\sqrt{2}}$, $\frac{dA}{dx}$ is -ve.

Thus $x = \frac{1}{\sqrt{2}}$ gives relative minimum. The dimensions of the largest inscribable rectangle are

$$\text{base} = 2x = \sqrt{2}, \text{ height} = y = \frac{1}{\sqrt{2}}$$

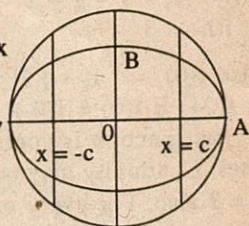
$$\text{Area } A = 1.$$

27. Required area is

$$4 \int_0^c \left[\sqrt{a^2 - x^2} - \frac{b}{a} \sqrt{a^2 - x^2} \right] dx$$

$$= 4 \left(1 - \frac{b}{a} \right) \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^c$$

$$= 4 \left(1 - \frac{b}{a} \right) \left(\frac{c}{2} \sqrt{a^2 - c^2} + \frac{a^2}{2} \sin^{-1} \frac{c}{a} \right)$$



28. $\int \frac{1 + \sin 2x}{\sin x (1 + \cos x)} dx$

$$= \int \frac{dx}{\sin x (1 + \cos x)} + 2 \int \frac{\cos x dx}{(1 + \cos x)}$$

$$= \frac{1}{2} \int \frac{1 + \tan^2 \frac{x}{2}}{\tan \frac{x}{2}} d\left(\tan \frac{x}{2}\right) + 2x - \int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2} \left[\ln \left| \tan \frac{x}{2} \right| + \frac{1}{2} \tan^2 \frac{x}{2} \right] + 2x - 2 \tan \frac{x}{2} + C$$

$$= 2x - 2 \tan \frac{x}{2} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{4} \tan^2 \frac{x}{2} + C$$

29. Put $x = 3 \tan u \Rightarrow du = 3 \sec^2 u \, du$

$$\int \frac{dx}{(x^2 + 9)^2} = \int \frac{3 \sec^2 u \, du}{81 \sec^4 u} = \frac{1}{27} \int \cos^2 u \, du$$

$$= \frac{1}{54} u \int (1 + \cos 2u) \, du$$

$$= \frac{1}{54} u + \frac{1}{108} \sin 2u + C$$

$$= \frac{1}{54} \tan^{-1} \frac{x}{3} + \frac{x}{18(x^2 + 9)} + C$$

30. $f(x) = |x| + |x - 1| + |x - 2|$

For $x < 0$, as well for $x > 2$, $f(x)$ is continuous and differentiable also.

At $x = 0$

Let $x = h > 0$ where $h < 1$

$$f(h) = h + 1 - h + 2 - h = 3 - h$$

$$\lim_{h \rightarrow 0^+} f(h) = 3$$

Let $x = h < 0$

$$f(h) = -h + 1 - h + 2 - h = 3 - 3h$$

$$\lim_{h \rightarrow 0^-} f(h) = 3$$

$$\text{Also } f(0) = |-1| + |-2| = 3$$

$$\therefore f(0^+) = f(0) = f(0^-)$$

$\therefore$ The function is continuous at $x = 0$ on similar lines. Continuity may be established for $x = 1$, for $x = 2$ also. For every other x also between 0 and 2 continuity is available.

Differentiability :-

At $x = 0$, $f(h)$ ($h > 0$) $= h + 1 - h + 2 - h = 3 - h$

$$f(0) = 3$$

$$\therefore \frac{f(h) - f(0)}{h} = \frac{3 - h - 3}{h}$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = -1 = f'(0^+)$$

$$\text{Let } f(h) \text{ } (h < 0) = -h + 1 - h + 2 - h = 3 - 3h$$

$$\text{In this } \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = -3 = f'(0^-).$$

$\therefore$ The R.H.S. derivative $\neq$ The L.H.S. derivative.

Hence $f(x)$ is NOT differentiable at $x = 0$. So, in fact NOT differentiable at $x = 1$ and $x = 2$; but differentiable everywhere else.

31. If possible let $\sqrt{3}$ be a rational number.

Then let $\sqrt{3} = p/q$, where p and q are integers prime to each other and $q \neq 0$. $\therefore p/q = \sqrt{3} \therefore p^2 = 3q^2$ or $p^2/q = 3q$.

This is not possible because the L.H.S. is a fraction but the R.H.S. is an integer. Hence the result.

$$32. S_n = 2 \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} \right]$$

$$= 2 \left(1 - \frac{1}{n+1} \right) \rightarrow 2 \quad \text{Ans. (c)}$$

33. Ans. (c). The given equation is

$$(|x| - 1)^3 - 1 = 0 \Rightarrow |x| - 1 = 1 \Rightarrow x = \pm 2$$

34. Ans. (b) The equations have only one (or a unique) solution when

$$\begin{vmatrix} k & 2 & -1 \\ 0 & k-1 & -2 \\ 0 & 0 & k+2 \end{vmatrix} \neq 0$$

and this does happen for $k = -1$, but not $k = -2, 0, 1$

35. Ans. (c)

$$\text{Product : } \cos \sum_{n=0}^4 \frac{2n\pi + 3\alpha}{5} + i \sin \sum_{n=0}^4 \frac{2n\pi + 3\alpha}{5}$$

$$\therefore \cos 3\alpha + i \sin 3\alpha$$

36. Ans. (d)

$$\cos \pi/2 = 0, \cos(-\theta) = \cos \theta$$

$$\cos(\pi + \theta) = -\cos \theta$$

37. Ans. (b) For $0 \leq x < 1$, $y = [x + 1] \Rightarrow y = 1$

$$\text{Area} = \int_0^1 (1 - x^2) dx = 2/3$$

38. False

The expression inside the root should be +ve with $x \neq -3$.

$$\therefore (2^2)^{\frac{3x^2+18x+29}{x+3}} \geq 2^{6x+17}$$

$$\Rightarrow \frac{6x^2+36x+58}{x+3} \geq 6x+17 \quad \dots(1)$$

If $x+3 > 0$, this becomes

$$6x^2 + 36x + 58 \geq 6x^2 + 35x + 51$$

(i.e) $x \geq -7$ already $x > -3$.

In this case, the admissible values are $x > -3$.

If $x+3 < 0$, inequality (1) becomes, $x \leq -7$ already $x < -3$.

The admissible values are $x \leq -7$

$\therefore$ The domain of definition is

$$x \in (-\infty, -7) \cup (-3, \infty)$$

$$39. \sin^{-1} \sqrt{\frac{2-\sqrt{3}}{4}} = \sin^{-1} \left(\frac{\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}}{2} \right)$$

$$= \sin^{-1} \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\pi}{12}$$

$$\therefore \cos^{-1} \frac{\sqrt{12}}{4} = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$\therefore \sec^{-1} \sqrt{2} = \frac{\pi}{4}$$

$$\text{So the value} = \sin^{-1} \left[\cot \frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4} \right]$$

$$= \sin^{-1}(\cot \pi/2)$$

$$= \sin^{-1}(0) = 0$$

40. Any one of the seven pictures can be hung from the first nail; and the one of the remaining 6 from the 2nd and so on, hence no. of ways

$$= 7 \times 6 \times 5 \times 4 \times 3 = 2520.$$



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CAREER FOR YOU

DAIRY INDUSTRY

With the delicensing of the dairy industry, entrepreneurs are increasingly rushing into this sector in order to make their pile by selling branded milk and value-added milk products. One reason for the bright prospects of the Indian dairy industry is the abundant supply of the relevant raw material - milk. Indian milk production increased by 4 percent in the last three years as against a decline of 2 percent in the world milk production, according to FAO estimates. In 1990, India lagged behind USSR and USA in milk production with 54.9 million tonnes; today it ranks second in the world (60.82 million tonnes), after the US (70 million tonnes). According to industry estimates, at the present rate of growth, India is expected to overtake the US by the year 2000.

This increase has been fuelled by better prices offered to farmers by new private companies, coupled with the adoption of good technology for production and distribution. The role of the organised sector is steadily growing in milk production as in processed milk. Anticipated reduction in production subsidies and import tariffs in Europe and the US should increase the competitiveness of India's dairy exports. The value of Indian dairy industry produce is expected to rise from Rs. 26,000 crore in 1990 to Rs. 47,500 crore in 1995 and reach Rs. 85,000 crore by 2000 A.D.

The scenario is just right for young people to aim for careers linked to the industry.

Dairy Technologists apply principles of bacteriology, chemistry, physics, engineering and economics to develop new and improved methods in production, preservation and utilisation of milk, cheese and other dairy products.

Dairy scientists study the physiology of reproduction and lactation and carry out breeding programmes to improve dairy breeds. They conduct experiments to determine effects of different kinds of feed and environmental conditions on quantity, quality and nutritive value of milk produced. They research breeding, feeding and management of dairy cattle. The work in the field of Dairy Science and Dairy Technology is of a scientific nature and hence persons who wish to enter this field must have an academic bent of mind with above average intelligence.

They would have to be interested in newer methods and improvements and continue to up-date their knowledge. Patience and perseverance to achieve the required results are helpful attributes here. Although the work is technical, one should have an interest both in the theoretical and practical applications of principles and techniques. Common sense and observation are important for the job.

Those who are into research should be able to work for long hours on their own. For those who work in the public or private sector organisations, the ability to handle workers and interact with colleagues and seniors are also important qualities.

When to embark on study and training

Dairy Technology is a four-year programme for which admission is granted on the basis of performance in competitive written exams conducted by the individual institutes/college.

Students with a minimum aggregate of 55 per cent marks in their Plus Two or equivalent exam in the Science stream with the subjects Physics, Chemistry, Maths and English or with Agriculture

are eligible for this course.

After graduating in Dairying one can opt for further studies and specialisation through the master's programme in Dairying. Disciplines for specialisation at the master's level include: Dairy Microbiology; Dairy Chemistry; Dairy Technology; Dairy Engineering; Genetics and Breeding; Livestock Production and Management, Animal Nutrition; Animal Physiology Biochemistry; Dairy Economics; Dairy Extension Education; Animal Biotechnology.

Admission to the postgraduate programme in Dairying is open to graduates in Pure Science/ Agriculture/Veterinary Sciences/Dairying/Food Technology/Engineering/Home Science with aggregate percentages above 55 (preferred 60 per cent and above). A competitive written exam is used to select students for the programmes at this level too.

Those who wish to pursue their Ph.D. in one of the fields of specialisation (as mentioned in the master's level course) should have a master's degree in the relevant area with 55-60 percent marks. Selection to the doctoral programmes is through a competitive entrance exam.

Important Institutions for Study and Training

- National Dairy Research Institute (deemed university); Karnal - 132001.

Courses : B.Tech (Dairy Technology), Master in Dairying, Ph.D. in Dairying. Minimum educational qualifications for all should be above 60 per cent

- Sheth M.C. College of Dairy Science, Anand Campus, Anand- 381 110.

All programmes are available here too.

- Andhra Pradesh Agricultural University, Rajendra Nagar, Hyderabad - 500 030.

- Dairy Science Institute, Aarey Milk colony (Government of Maharashtra), Bombay - 400 064 and its regional centres at:

Bangalore in Karnataka ; District Nadia in West Bengal; Allahabad in UP.

Course : A two-year Indian Dairy Diploma in Dairy

Technology/Dairy Husbandry.

Eligibility : Plus Two or equivalent exam with at least 50 percent marks in the aggregate with the subjects Physics, Chemistry, Maths and English. A joint entrance exam is held for all the four regions. Boarding is compulsory for the programme.

- National Dairy Research Institute, Southern Regional Station, Bangalore.

Course : Diploma in the Management of Dairy Enterprises.

Eligibility : Graduates in Dairy Technology/Dairying/Animal Husbandry/Agriculture.

- Indian Institute of Management, Bangalore. Graduates in Dairy Technology/Veterinary Sciences/ Agriculture or any related field are also eligible to join here for a postgraduate programme in Agriculture and Rural Development.

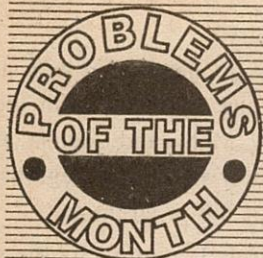
- The Institute of Rural Management, Anand. *Course :* Postgraduate programme in Rural Management, which equips personnel to manage modern processing units and market consumer goods for producer cooperatives.

Eligibility : Graduates who must qualify in the written test for admission and succeed in the group discussion and interview which follow.

Placements

Dairy Technologists/Scientists/Managers find placements in public sector organisations such as the National Dairy Development Board (NDDB), dairy cooperative societies and dairy plants manufacturing milk products in the different states. Almost all states now have their own dairy producers cooperatives which undertake the manufacture, production and marketing of dairy products such as ghee, liquid milk, cheese, yogurt, butter, powder milk, condensed milk and other processed milk products.

The private dairy industry also offers great scope as delicensing has led to a flood of new entrants. Organisations in this sector include milkfood, Dalmia industries, Nestle, Vadilal, Amrut Industries, Heritage Foods, Indana Dairy specialities, J.K. ■



VECTORS

— Anuman Banerjee, Dhakuria
Calcutta

1. A straight line meets the sides BC, CA and BA produced of triangle ABC at L, M and N respectively. Hence, prove vectorially

$$\left(\frac{BL}{CL}\right) \times \left(\frac{CM}{AM}\right) \times \left(\frac{AN}{BN}\right) = 1$$

2. Show that the solution of the equation $k\vec{r} + \vec{r} \times \vec{a} = \vec{b}$ where k is a nonzero scalar and $\vec{a}$ and $\vec{b}$ are vectors is of the form

$$\vec{r} = \frac{1}{(k^2 + a^2)} \left[\frac{\vec{a} \cdot \vec{b}}{k} \cdot \vec{a} + k\vec{b} + \vec{a} \times \vec{b} \right]$$

3. Show that the shortest distance between a diagonal of a rectangular parallelepiped the lengths of whose three coterminal edges are a, b, c and the edges not meeting are $\frac{bc}{\sqrt{b^2 + c^2}}, \frac{ca}{\sqrt{c^2 + a^2}}, \frac{ab}{\sqrt{a^2 + b^2}}$.

4. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$; $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and

$\vec{b}$. If angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, find $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$

5. If $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}, x_2\vec{a} + y_2\vec{b} + z_2\vec{c}, x_3\vec{a} + y_3\vec{b} + z_3\vec{c}, x_4\vec{a} + y_4\vec{b} + z_4\vec{c}$ are four coplanar vectors, show

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} = 0, \text{ where } \vec{a}, \vec{b}, \vec{c} \text{ are noncoplanar.}$$

6. ABC is any triangle and O is any point in its plane inside it. AO, BO and CO meet BC, CA and

AB in D, E, F respectively. Prove vectorially

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1.$$

7. A line makes angle $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$. Also show that the angle between the two diagonals of the cube is $\cos^{-1}(1/3)$.

8. Find the equation of the plane passing through $(3, -2, 1)$ and perpendicular to $4\hat{i} + 7\hat{j} - 4\hat{k}$. Let BM be the perpendicular from $B(1, 2, -1)$ to this plane. Find the length of BM.

9. $a\hat{i} + \hat{j} + \hat{k}, \hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are three coplanar vectors. ($a \neq b \neq c \neq 1$) Find

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}.$$

10. Let ABC be a triangle with $AB = AC$. If D is midpoint of BC, E be foot of perpendicular from D on AC and F is midpoint of DE. Prove AF and BE are at right angles.

11. A rigid body is rotating at 2.5 rad s^{-1} about an axis AR where A and R are respectively the points $\hat{i} - 2\hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} + 2\hat{k}$ relative to an origin. Find velocity of P at point $5\hat{i} - \hat{j} - \hat{k}$.

12. Determine a set of equations for the straight lines passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

13. Graph the vector field defined by $V(x, y) = x\hat{i} + y\hat{j}$.

14. $\vec{A} = 2\hat{i} + \hat{k}, \vec{B} = \hat{i} + \hat{j} + \hat{k}, \vec{C} = 4\hat{i} - 2\hat{j} + 7\hat{k}$.

Determine $\vec{R}$ such that the vector R satisfies
 $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$

15. Let ABC and PQR be two triangles. Perpendiculars drawn from A to QR, B to RP, C to PQ are concurrent. Prove that the perpendicular from P to BC, Q to CA, R to AB are also concurrent.

16. Let P_1, P_2, P_3 be points fixed relative to an origin O and let r_1, r_2, r_3 be position vectors from O to each point. Show that if the vector equation $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$ holds with respect to origin O, then it will hold with respect to any other origin O' if and only if $a_1 + a_2 + a_3 = 0$.

17. Consider a tetrahedron with faces F_1, F_2, F_3, F_4 . Let V_1, V_2, V_3, V_4 be vectors whose magnitudes are respectively equal to the areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to these faces in the outward direction. Show $V_1 + V_2 + V_3 + V_4 = 0$.

18. Consider the nonzero vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that no three are coplanar. Then show

$$\vec{a}[\vec{b} \vec{c} \vec{d}] + \vec{c}[\vec{a} \vec{b} \vec{d}] = \vec{b}[\vec{a} \vec{c} \vec{d}] + \vec{d}[\vec{a} \vec{b} \vec{c}].$$

Hence prove if $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ represent the position vectors of the vertices of a plane quadrilateral if

$$\frac{[\vec{b} \vec{c} \vec{d}]}{[\vec{a} \vec{c} \vec{d}]} + \frac{[\vec{a} \vec{b} \vec{d}]}{[\vec{a} \vec{b} \vec{c}]} = 1.$$

SOLUTIONS

1. Let $\frac{BL}{CL} = \frac{m}{n}, \frac{CM}{AM} = \frac{p}{q}, \frac{AN}{BN} = \frac{r}{s}$

Let $\vec{BC} = \vec{c}, \vec{BA} = \vec{a}$

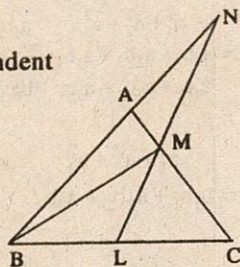
$\vec{c}$ and $\vec{a}$ are linearly independent

$$\therefore \frac{BL}{CL} = \frac{m}{n}$$

$$\therefore \vec{BL} = \frac{m}{m+n}(\vec{BL} + \vec{CL})$$

$$= \frac{m}{m+n} \vec{BC} = \frac{m}{m+n} \vec{c}$$

Similarly, $\vec{BM} = \frac{p\vec{a} + q\vec{c}}{p+q}$



$$\frac{AN}{BN} = \frac{r}{s}$$

$$\text{or } \frac{BN - AB}{BN} = \frac{r}{s}$$

$$\text{or } \vec{BN} = \frac{s}{s-r} \vec{BA} = \frac{s}{s-r} \vec{a}$$

Since L, M, N are collinear, there exists non zero scalars λ, μ, γ such that

$$\lambda \vec{BL} + \mu \vec{BM} + \gamma \vec{BN} = 0$$

$$\text{where } \lambda + \mu + \gamma = 0$$

$$\therefore \lambda \vec{BL} + \mu \vec{BM} + \gamma \vec{BN} = 0$$

$$\text{or, } \lambda \frac{m\vec{c}}{m+n} + \mu \frac{p\vec{a} + q\vec{c}}{p+q} + \gamma \frac{s}{s-r} \vec{a} = 0$$

$$\text{or, } \vec{c} \left(\frac{\lambda m}{m+n} + \frac{\mu q}{p+q} \right) + \vec{a} \left(\frac{\mu p}{p+q} + \frac{\gamma s}{s-r} \right) = 0$$

$\therefore \vec{c}$ and $\vec{a}$ are linearly independent.

$$\therefore \frac{\lambda m}{m+n} + \frac{\mu q}{p+q} = 0 \quad \dots (1)$$

$$\text{and } \frac{\mu p}{p+q} + \frac{\gamma s}{s-r} = 0 \dots (2)$$

$$\therefore \lambda + \mu + \gamma = 0$$

$$\frac{\lambda}{\mu} + 1 + \frac{\gamma}{\mu} = 0$$

$$\text{or, } \frac{-q}{p+q} \times \frac{m+n}{m} + 1 + \frac{p}{p+q} \times \frac{r-s}{s} = 0$$

$$\text{or, } \frac{-qs(m+n) + ms(p+q) + mp(r-s)}{ms(p+q)} = 0$$

$$\text{or, } -mqs - nqs + mps + mqs + mpr - mps = 0$$

$$\text{or, } mpr = nqs$$

$$\text{or, } \frac{m}{n} \times \frac{p}{q} \times \frac{r}{s} = 1$$

$$\text{or, } \frac{BL}{CL} \times \frac{CM}{AM} \times \frac{AN}{BN} = 1$$

Proved.

2. $\vec{b} = k\vec{r} + \vec{r} \times \vec{a}$

$$\vec{a} \cdot \vec{b} = k\vec{r} \cdot \vec{a} + \vec{a} \cdot (\vec{r} \times \vec{a})$$

$$= k(\vec{r} \cdot \vec{a}) + [\vec{a} \vec{r} \vec{a}]$$

$$= k(\vec{r} \cdot \vec{a}) + 0 = k(\vec{r} \cdot \vec{a}).$$

$$\vec{a} \times \vec{b} = k(\vec{a} \times \vec{r}) + \vec{a} \times (\vec{r} \times \vec{a})$$

$$= k(\vec{a} \times \vec{r}) + (\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{r})\vec{a}$$

$$k\vec{b} = k\vec{r} + k(\vec{r} \times \vec{a})$$

$$\therefore \frac{\vec{a} \cdot \vec{b}}{k} (\vec{a}) = \frac{k(\vec{r} \cdot \vec{a})}{k} \times (\vec{a}) = (\vec{r} \cdot \vec{a}) \vec{a}$$

$$\therefore \vec{a} \times \vec{b} + k\vec{b} + \frac{\vec{a} \cdot \vec{b}}{k} (\vec{a}) = k(\vec{a} \times \vec{r}) + a^2\vec{r} -$$

$$(a \cdot \vec{r})\vec{a} + k\vec{r} + k(\vec{r} \times \vec{a}) + (\vec{r} \cdot \vec{a})\vec{a}$$

$$= k(\vec{a} \times \vec{r}) + a^2\vec{r} - (\vec{r} \cdot \vec{a})\vec{a} + k\vec{r} - k(\vec{a} \times \vec{r}) + (\vec{r} \cdot \vec{a})\vec{a}$$

$$= (k^2 + a^2)\vec{r}$$

$$\therefore \vec{r} = \frac{1}{(k^2 + a^2)} \left[\frac{\vec{a} \cdot \vec{b}}{k} (\vec{a}) + \vec{a} \times \vec{b} + k\vec{b} \right]$$

3. Let a, b, c be the lengths of the sides OA, OB and OC respectively of the rectangular parallelepiped taking O as origin of vectors, let $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along OA, OB, OC respectively.

$$\vec{OA} = a\hat{i}$$

$$\vec{OB} = b\hat{j}$$

$$\vec{OC} = c\hat{k}$$

$$\text{Also } \vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$$

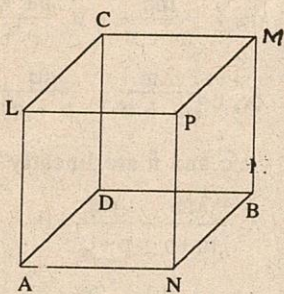
The edges which do not meet the diagonals OP are AL, AN, BN and their parallels BM, CM and CL . We are to find the distance between the diagonal OP and the edge BN .

Now OP is passing through O which is parallel to the vectors $a\hat{i} + b\hat{j} + c\hat{k}$. BN is the line through B whose position vector is $b\hat{j}$. The shortest distance between two nonintersecting lines passing through the points whose position vectors are a and b and are parallel to $\vec{c}$ and $\vec{d}$ is

$$p = \frac{(\vec{a} - \vec{b}) \cdot (\vec{c} \times \vec{d})}{|\vec{c} \times \vec{d}|}$$

$$= \frac{[\vec{a} - \vec{b}, \vec{c}, \vec{d}]}{|\vec{c} \times \vec{d}|}$$

The shortest distance between OP and BN is given by



$$p_1 = \frac{[0 - b\hat{j}, \hat{i}, a\hat{i} + b\hat{j} + c\hat{k}]}{|\hat{i} \times (a\hat{i} + b\hat{j} + c\hat{k})|}$$

$$\text{Now, } [0 - b\hat{j}, \hat{i}, a\hat{i} + b\hat{j} + c\hat{k}] = bc$$

$$\hat{i} \times (a\hat{i} + b\hat{j} + c\hat{k}) = b\hat{k} - c\hat{j}$$

$$\therefore |\hat{i} \times (a\hat{i} + b\hat{j} + c\hat{k})| = \sqrt{b^2 + c^2}$$

$$\therefore p_1 = \frac{bc}{\sqrt{b^2 + c^2}}$$

Similarly, shortest distance OP and AL is $\frac{ab}{\sqrt{a^2 + b^2}}$

and that between OP and AN is $\frac{ca}{\sqrt{c^2 + a^2}}$.

4. According to the problem,

$$c_1^2 + c_2^2 + c_3^2 = 1$$

$$\vec{a} \cdot \vec{c} = 0, \vec{b} \cdot \vec{c} = 0 \text{ and}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$\therefore \vec{a} \cdot \vec{c} = 0 \therefore a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$\therefore \vec{b} \cdot \vec{c} = 0 \therefore b_1c_1 + b_2c_2 + b_3c_3 = 0$$

$$\frac{\sqrt{3}}{2} [(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)]^{\frac{1}{2}} = a_1b_1 + a_2b_2 + a_3b_3$$

Now,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ a_1c_1 + a_2c_2 + a_3c_3 & b_1c_1 + b_2c_2 + b_3c_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & 0 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$= \frac{3}{4} (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Use More Mind Power

for Engineering Entrance Exams

Dear friend,

My name is Raj Bapna. In this letter, I have something very important to say that can help you greatly to get success in your exams and competitions. If you have 10 minutes, I request you to read this page about how to study, how to use more mind power, how to improve your memory and much more.

Yogis have always known it and scientists have also discovered it now—that most people use only 10% of their mind power.

Improve Your Memory Quickly

Of many easy techniques, I explain two:

ONE. The brain has two memory stores: short-term and long-term. Research shows that without revision, after 24 hours we remember only 18%. After 1 month only 5%. It clearly shows that we must revise. But, most students do not revise systematically, so much of their hard work is wasted. I teach you the powerful techniques "Systematic Revision" and "Daily Routine" so that you can revise and remember more in less time.

TWO. Scientific research has proved that for better memory, we should take rest and not study continuously for hours. You will learn my technique "Rest Routine" to get maximum benefit from the rest. This technique relaxes you, changes your brain waves, and puts you in a "learning state".

New All India Memory Record

One of our students R Caudally has recently set All India Memory Record. In interviews to many newspapers he said "The secret of my newly developed memory are postal courses **Mind Power Music** and **Mind Power Study Techniques** from the Mind Power Research Institute, Udaipur." Before joining our courses, he was an average student and scored only 52.3% in High School Exam.

Are you thinking now "If my course can help someone to set a new memory record, can it help me to get more success in your exam?"

Read Faster to Revise Faster

Everyone can learn to read and understand 300, or 500 or more words per minute. But, many of us read only about 100 words per minute. My "Finger Technique" will double your reading speed in 30 minutes. Your read slow if:

- If you read aloud or move lips
- If you do not read aloud but hear the sounds in your mind when you read
- If you read one word at a glance rather than reading many words at a glance
- If, without your knowledge, you read some words again and again.

My course will help you to overcome these habits. The best use of reading faster is not to study new chapters for the first time, but to revise again and again quickly so that you can remember more in less time. The "Finger Technique" helped me to increase my reading speed from 72 to 1037 words per minute. Here is what two experts say about this technique:

"I am very happy to inform you that my son Ravi Anand increased his reading speed from 228 to surprisingly high 1818 words per minute. Thank you for your course."

—Dr M L Singh, MBBS, MS, Eye Surgeon, Bihar

"Unbelievably, I improved my reading speed from 75 to 200 words per minute. My son improved his memory. He also improved his reading speed from 45 to 100."

—Prof M Bhattacharya, PhD, Formerly in USA

Simple, Practical, Effective

My techniques are effective. They do not make you tired. And you can learn them fast. I teach no theory. Only the techniques that have proved effective for myself and other students.

You may find it difficult to fully understand the power and benefits of my course just by reading

this page now. But, those who join my course will benefit greatly and avoid mistakes that can cause failure for others.

The newspaper **Times of India**, says that from my course you learn "Simple, effective, practical techniques to improve overall intelligence and mind power. Even average student can easily understand."

How Will You Greatly Benefit

Before you read this page fully, I want to make it clear that my course cannot give success by magic. But with my course, you can be more sure of success because you become better than 99% of students in the following **9 Critical Success Factors**:

1. Good increase in your memory and concentration
2. Your effectiveness to read and learn will increase greatly
3. Your ability to study longer without getting tired (body or mind) or feeling sleepy will increase
4. You will experience that you are capable of achieving much more success than you currently do (even if you are already very good)
5. Small to moderate improvement in your intelligence
6. Set realistically high aims/goals and take you step by step on the road to achieve success
7. Improve writing, spelling, interview skills
8. Learn exam secrets to get more marks for what you have studied
9. Avoid big mistakes that can result in failure.

Suppose you improve only 5% in each, then total improvement is $5 \times 9 = 45\%$. I know you will improve 100% just in reading speed. So, total improvement will be great for your success.

I Was Not Always Successful

I want you to know that I was not always highly successful as a student. You can call it luck or chance that I happened to discover a few techniques to study for success. These techniques changed my life and my marks improved in three years from 73.0% to 78.0%, 83.5%, 87.7%. Similarly, I did not get NTS scholarship in class 10 because I made a simple, stupid mistake. Then in class 11, I did not make the mistake and I got success in NTS.

Do you realize that if just a few techniques improved my success so much, what my complete course can do for your success?

What is Unique About It

My course combines 5000 year old Indian techniques with the latest scientific discoveries in brain research, nutrition, psychology, and music in America and other countries.

In USA, just before returning to India, I spent 1300 dollars (about Rs 42,000) to join two courses to learn 3 more mind power techniques. You will learn them in my course. My personal library has books and courses worth Rs 1,17,210. I have read, experimented, researched with all their techniques and included only the best ones into my course. These techniques are in addition to my own developed techniques in the course. This course is protected by the Copyright Law, and so nobody can copy my material.

Used by Lakhs World-Wide

Lakhs of people from every corner of India and from many parts of the world are benefiting everyday from my course. Consider just this simple fact: If a course from India is used even abroad, the course must be really good.

Do you understand fully that you can decide to order this course now to help you to get success and also to fulfil your parent's hopes and dreams?

Music for Success

It has powerful effects on your mind/brain. So, it is not for people with epilepsy, and anyone undergoing psychiatric or electro-therapy.

It is based on scientific research into how the mind works and how to program and control it for

BIO-DATA

- B.E., BITS Pilani. M.Tech., IIT Kharagpur. NTSE scholar. Rank 5 Raj School Board.
- World-famous author. I published 3 computer books in USA. One is best selling (**MS-DOS Masters**).
- Increased my reading speed from 72 to 1037 words per minute. Was a member of **Society for Accelerated Learning & Teaching, USA**.
- My first job as an engineer paid only Rs 1000 per month. Just 7 years later, I earned 50 dollars per HOUR in USA as computer expert.
- At the peak of success, I returned to India to do something in our own country. Now, I spend my full time as a scientist to do mind power research.
- I also learnt French, Sanskrit, Karate, Breaking wooden board by hand, many Meditations, etc.

our own success. It has sounds from instruments and nature (river, birds). For details on how such relaxing music helps to learn faster, please read USA best-seller books "Superlearning" and "MegaBrain".

The Hidden-Messages™ in music bypass your conscious mind and go to your subconscious mind, and change your behaviour. Here is what people say:

"I have already purchased a course of Mind Power Music. Please send me 6 more for the use of my staff. Thank you."

—Rector (Principal), Holy Rock School, Burdwan, W.B.

"Very good. It relaxes my body and mind. It reduces the tension of my studies."

—Dr Anju Banthia, MBBS, Bhopal

Money-Back GUARANTEE

Order my course (code 805 or 712) and if you are not 100% satisfied, tear it into 2 pieces and return in 31 days, and I will send your money (less Rs 30 for processing charge). No questions will be asked. If you order this month, I will also send a poster of **Bapna's Optical Illusion TM Technique for Concentration**. Keep it as **free gift** even if you return the course.

Time Does Not Wait for Anyone

It is now up to you. You can turn this page as if you did not even read it and miss this opportunity for more success. Or, you can join this course today. Will the coming weeks and months make you a much better student by joining this course? Or, will you remain like many others and struggle for success?

You decide.

Discount Prices

Course Name	Course Code	Price + Post
Mind Power		
Mind Power Study Techniques course	805	145+15
Memory and Concentration cassette (FREE: get book Mind Power Music)	110	66+10
Engineering Subjects		
Memory Maps for Physics	510	225+15
Memory Course for Chemistry (FREE: get book Advanced Numericals and Questions and Answers in Chemistry)	521	240+15
Memory Course for Zoology	531	235+15
Memory Course for Botany	541	235+15
Memory Course for Maths	551	385+15

Combined Offers: Save Even More

course name	code	price	combined price+post
Medical 4 courses (511, 521, 531, 541)	911	935+60	675+15
Engineering 3 courses (510, 521, 551)	922	850+45	675+15

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This institute is now the world's largest mind power institute for students

$$= \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

5. Let $\vec{A} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$

$$\vec{B} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$$

$$\vec{C} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$$

$$\vec{D} = x_4\vec{a} + y_4\vec{b} + z_4\vec{c}$$

Since $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are coplanar, there exist scalars $\alpha, \beta, \gamma, \phi$ such that

$$\alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} + \phi\vec{D} = 0 \quad \dots(1)$$

$$\text{where } \alpha + \beta + \gamma + \phi = 0 \quad \dots(2)$$

Again, we have if $x\vec{p} + y\vec{q} + z\vec{r} = 0$ where $\vec{p}, \vec{q}, \vec{r}$ are noncoplanar, then $x = y = z = 0$

$$\text{Here, } \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} + \phi\vec{D} = 0$$

$$\text{or, } \alpha(x_1\vec{a} + y_1\vec{b} + z_1\vec{c}) + \beta(x_2\vec{a} + y_2\vec{b} + z_2\vec{c}) + \gamma(x_3\vec{a} + y_3\vec{b} + z_3\vec{c}) + \phi(x_4\vec{a} + y_4\vec{b} + z_4\vec{c}) = 0$$

$$\text{or, } \vec{a}[\alpha x_1 + \beta x_2 + \gamma x_3 + \phi x_4] + \vec{b}[\alpha y_1 + \beta y_2 + \gamma y_3 + \phi y_4] + \vec{c}[\alpha z_1 + \beta z_2 + \gamma z_3 + \phi z_4] = 0$$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are noncoplanar

$$\therefore \alpha x_1 + \beta x_2 + \gamma x_3 + \phi x_4 = 0$$

$$\alpha y_1 + \beta y_2 + \gamma y_3 + \phi y_4 = 0$$

$$\alpha z_1 + \beta z_2 + \gamma z_3 + \phi z_4 = 0$$

So we have four equations in $\alpha, \beta, \gamma, \phi$:-

$$\alpha + \beta + \gamma + \phi = 0$$

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \phi x_4 = 0$$

$$\alpha y_1 + \beta y_2 + \gamma y_3 + \phi y_4 = 0$$

$$\alpha z_1 + \beta z_2 + \gamma z_3 + \phi z_4 = 0$$

Eliminating,

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} = 0$$

6. Let O be the origin of reference

$$\text{Let } \vec{OA} = \vec{a}$$

$$\vec{OB} = \vec{b}$$

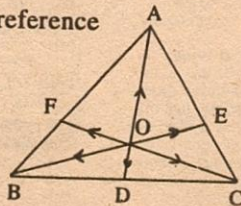
$$\vec{OC} = \vec{c}$$

since $\vec{OA}, \vec{OB}, \vec{OC}$ are

coplanar, there exists

nonzero scalars such that

$$\lambda\vec{a} + \mu\vec{b} + \gamma\vec{c} = 0 \text{ where } \lambda, \mu, \gamma \text{ are nonzero scalars.}$$



$$\text{Now, } \vec{OD} = -t \vec{OA} \quad (t > 0)$$

$$= -t\vec{a} = +\frac{t}{\lambda}(\mu\vec{b} + \gamma\vec{c})$$

$$\text{Now, } \frac{BD}{DC} = \frac{m}{n} \text{ (say)}$$

$$\vec{OD} = \frac{m\vec{c} + n\vec{b}}{m+n}$$

$$\therefore \frac{t}{\lambda} \mu\vec{b} + \frac{t}{\lambda} \gamma\vec{c} = \frac{m}{m+n} \vec{c} + \frac{n}{m+n} \vec{b}$$

$$\text{or, } \left(\frac{t}{\lambda} \mu - \frac{n}{m+n}\right) \vec{b} + \left(\frac{t}{\lambda} \gamma - \frac{m}{m+n}\right) \vec{c} = 0$$

Since $\vec{b}$ and $\vec{c}$ are linearly independent,

$$\frac{t}{\lambda} \mu = \frac{n}{m+n}, \quad \frac{t\gamma}{\lambda} = \frac{m}{m+n}$$

$$\therefore \frac{t}{\lambda} (\mu + \gamma) = \frac{n+m}{m+n} \Rightarrow t = \frac{\lambda}{\mu + \gamma}$$

$$t + 1 = \frac{\lambda + \mu + \gamma}{\mu + \gamma}$$

$$\frac{t}{t+1} = \frac{\lambda}{\lambda + \mu + \gamma}$$

$$\text{Again, } \frac{AO}{OD} = \frac{1}{t}$$

$$\text{or, } \frac{AO + OD}{OD} = \frac{1+t}{t}$$

$$\text{or, } \frac{AD}{OD} = \frac{1+t}{t}$$

$$\therefore \frac{OD}{AD} = \frac{t}{1+t} = \frac{\lambda}{\lambda + \mu + \gamma}$$

$$\text{Similarly, } \frac{OE}{BE} = \frac{\mu}{\lambda + \mu + \gamma}, \quad \frac{OF}{CF} = \frac{\gamma}{\lambda + \mu + \gamma}$$

(from symmetry of result).

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = \frac{\lambda}{\lambda + \mu + \gamma} + \frac{\mu}{\lambda + \mu + \gamma} + \frac{\gamma}{\lambda + \mu + \gamma} = 1$$

7. Let a be the edge of the cube so that

$$\vec{OA} = a\hat{i}$$

$$\vec{OB} = a\hat{j}$$

$$\vec{OC} = a\hat{k}$$

$$\vec{OD} = a(\hat{i} + \hat{j})$$

$$\vec{OE} = a(\hat{j} + \hat{k})$$

$$\vec{OF} = a(\hat{k} + \hat{i})$$

$$\text{The four diagonals are}$$

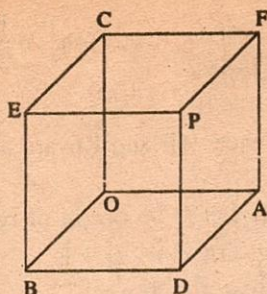
$$\vec{OP} = a(\hat{i} + \hat{j} + \hat{k})$$

$$\vec{CD} = a(\hat{i} + \hat{j} - \hat{k})$$

$$\vec{AE} = a(\hat{j} + \hat{k} - \hat{i})$$

$$\vec{CF} = a(\hat{k} + \hat{j} - \hat{i})$$

Let $L(x, y, z)$ be a point on the line drawn through O parallel to the given line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals.



$$\vec{OL} = x\hat{i} + y\hat{j} + z\hat{k},$$

$$|\vec{OL}| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \vec{OP} \cdot \vec{OL} = |\vec{OP}| |\vec{OL}| \cos \alpha$$

$$= |a(\hat{i} + \hat{j} + \hat{k})| |(x\hat{i} + y\hat{j} + z\hat{k})| \cos \alpha$$

$$\text{or, } a(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \sqrt{3}(x^2 + y^2 + z^2)^{1/2} a \cos \alpha$$

$$\text{or, } a(x + y + z) = \sqrt{3}(\sum x^2)^{1/2} a \cos \alpha$$

$$\therefore \cos \alpha = \frac{x + y + z}{\sqrt{3}(\sum x^2)^{1/2}}$$

Similarly, taking dot product of $\vec{OL}$ with the vectors represented by the other diagonals, we have

$$\cos \beta = \frac{x + y - z}{\sqrt{3}(\sum x^2)^{1/2}}, \quad \cos \gamma = \frac{y + z - x}{\sqrt{3}(\sum x^2)^{1/2}}$$

$$\cos \delta = \frac{z + x - y}{\sqrt{3}(\sum x^2)^{1/2}}$$

$$\begin{aligned} \therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta &= \frac{(x + y + z)^2 + (x + y - z)^2 + (y + z - x)^2 + (z + x - y)^2}{3(x^2 + y^2 + z^2)} \\ &= \frac{4}{3} \text{ (expanding numerator)} \end{aligned}$$

Again if angle between $\vec{OP}$ and $\vec{CD}$ be θ

$$\vec{OP} \cdot \vec{CD} = |\vec{OP}| |\vec{CD}| \cos \theta$$

$$\Rightarrow a(\hat{i} + \hat{j} + \hat{k}) \cdot a(\hat{i} + \hat{j} - \hat{k}) = (a\sqrt{3})(a\sqrt{3}) \cos \theta$$

$$\Rightarrow a^2 = 3a^2 \cos \theta$$

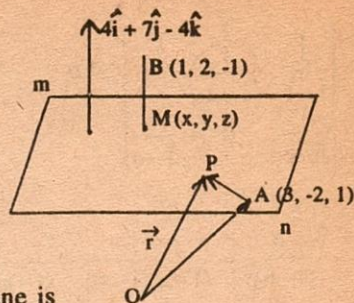
$$\Rightarrow \theta = \cos^{-1}(1/3)$$

8. Let P be any point on the plane mn . Let O be origin, $\vec{OP} = \vec{r}$

$$\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - 3\hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{AP} \cdot (4\hat{i} + 7\hat{j} - 4\hat{k}) = 0$$

$\therefore 4\hat{i} + 7\hat{j} - 4\hat{k}$ is perpendicular to mn and hence perpendicular to $\vec{AP}$ lying in mn



Eqn. to the plane is

$$(\vec{r} - 3\hat{i} + 2\hat{j} - \hat{k}) \cdot (4\hat{i} + 7\hat{j} - 4\hat{k}) = 0$$

$$\text{or, } \vec{r} \cdot (4\hat{i} + 7\hat{j} - 4\hat{k}) + 6 = 0$$

$$\text{Let } |\vec{BM}| = l$$

$$\therefore \vec{BM} = l \frac{(4\hat{i} + 7\hat{j} - 4\hat{k})}{|4\hat{i} + 7\hat{j} - 4\hat{k}|} = \frac{l(4\hat{i} + 7\hat{j} - 4\hat{k})}{9}$$

$\left[\frac{(4\hat{i} + 7\hat{j} - 4\hat{k})}{|4\hat{i} + 7\hat{j} - 4\hat{k}|} \right]$ is unit vector in direction of $\vec{BM}$ as $\vec{BM}$ is parallel to $4\hat{i} + 7\hat{j} - 4\hat{k}$

$$\text{or, } \vec{BM} = \frac{4l}{9}\hat{i} + \frac{7l}{9}\hat{j} - \frac{4l}{9}\hat{k}$$

$$\vec{OB} = \hat{i} + 2\hat{j} - \hat{k} \quad \vec{OM} = \vec{OB} + \vec{BM}$$

$$= \left(1 + \frac{4l}{9}\right)\hat{i} + \left(2 + \frac{7l}{9}\right)\hat{j} + \left(-1 - \frac{4l}{9}\right)\hat{k}$$

Now, M is a point on the plane mn .

$$\therefore \vec{OM} \text{ satisfies } \vec{r}.$$

$$\therefore \left[\left(1 + \frac{4l}{9}\right)\hat{i} + \left(2 + \frac{7l}{9}\right)\hat{j} - \left(1 + \frac{4l}{9}\right)\hat{k} \right] \cdot [4\hat{i} + 7\hat{j} - 4\hat{k}] = 0$$

$$\text{or, } |l| = 28/9 \text{ on usual calculation}$$

$$\therefore \text{Length of } BM \text{ is } 28/9 \text{ units or } 3 \frac{1}{9} \text{ units.}$$

9. Since the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar,

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\text{or, } \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0 \quad \begin{matrix} [R_2' \rightarrow R_2 - R_1] \\ [R_3' \rightarrow R_3 - R_1] \end{matrix}$$

$$\text{or, } (1-a)^2(b-1)(c-1) \begin{vmatrix} a & 1 & 1 \\ b-1 & 1-a & 0 \\ c-1 & 0 & 1-a \end{vmatrix} = 0$$

$$\text{or, } \begin{vmatrix} -a & 1 & 1 \\ 1-b & 1-a & 0 \\ 1 & 0 & 1-a \end{vmatrix} = 0.$$

[$\because a \neq b \neq c \neq 1$]

$$\text{or, } -\frac{a}{(1-a)^2} - \frac{1}{(1-a)(1-b)} - \frac{1}{(1-a)(1-c)} = 0$$

$$\text{or, } \frac{a}{(1-a)^2} + \frac{1}{(1-a)(1-b)} + \frac{1}{(1-a)(1-c)} = 0$$

$$\text{or, } \frac{a}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 0$$

$$\text{or, } \frac{1-(1-a)}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 0$$

$$\text{or, } \frac{1}{(1-a)} - 1 + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 0$$

$$\text{or, } \frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 1$$

10. Let $\vec{AB} = \vec{b}$

$\vec{AC} = \vec{c}$

$\vec{AD} = \frac{1}{2}(\vec{b} + \vec{c})$

Now, $\vec{AE} = \frac{\vec{AD} \cdot \vec{AC}}{|\vec{AC}|} \hat{n}$

$$= \frac{\frac{1}{2}(\vec{b} + \vec{c}) \cdot \vec{c}}{|\vec{c}|} \hat{n}$$

$$= \frac{\vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}}{2|\vec{c}|} + \frac{\vec{c}}{|\vec{c}|}$$

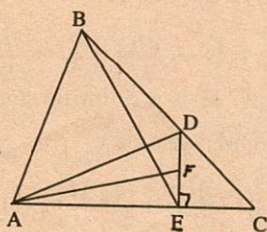
$$= \frac{(\vec{c} \cdot \vec{b}) \vec{c}}{2|\vec{c}|^2} + \frac{\vec{c}}{2}$$

$$\vec{AF} = \frac{1}{2}(\vec{AD} + \vec{AE}) = \frac{1}{2} \left\{ \frac{1}{2} \vec{b} + \vec{c} + \frac{(\vec{b} \cdot \vec{c}) \vec{c}}{2|\vec{c}|^2} \right\}$$

$$\vec{BE} = \vec{AE} - \vec{AB} = \frac{(\vec{b} \cdot \vec{c}) \vec{c}}{2|\vec{c}|^2} + \frac{1}{2} \vec{c} - \vec{b}$$

$$= \frac{1}{2} \left\{ \frac{(\vec{b} \cdot \vec{c}) \vec{c}}{|\vec{c}|^2} + \vec{c} - 2\vec{b} \right\}$$

$$\vec{AF} \cdot \vec{BE} = \frac{1}{4} \left\{ \frac{(\vec{b} \cdot \vec{c})^2}{2|\vec{c}|^2} + \vec{b} \cdot \vec{c} + \frac{(\vec{b} \cdot \vec{c})^2}{2|\vec{c}|^2} + \frac{1}{2}(\vec{b} \cdot \vec{c}) \right\}$$



$$+|\vec{c}|^2 + \frac{1}{2}(\vec{b} \cdot \vec{c}) - |\vec{b}|^2 - 2\vec{b} \cdot \vec{c} - \frac{(\vec{b} \cdot \vec{c})}{|\vec{c}|^2} \}$$

$$= 0$$

Hence, AF and BE are at right angles.

11. Let O be origin of reference

Let $\vec{OA} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{OR} = 3\hat{i} - 4\hat{j} + 2\hat{k}$

$\vec{OP} = 5\hat{i} - \hat{j} - \hat{k}$

Now, linear velocity

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where ω is angular velocity and

$$\vec{r} = \vec{AP}$$

$$= \vec{OP} - \vec{OA} = 4\hat{i} + \hat{j} - 2\hat{k}$$

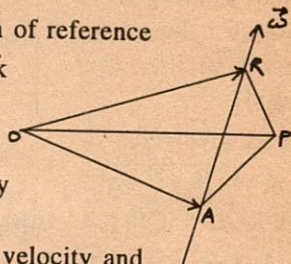
$$\vec{\omega} = 2.5 \times \frac{\vec{AR}}{|\vec{AR}|} = 2.5 \times \frac{(\vec{OR} - \vec{OA})}{|\vec{OR} - \vec{OA}|}$$

$$= 2.5 \times \frac{2\hat{i} - 2\hat{j} + \hat{k}}{|2\hat{i} - 2\hat{j} + \hat{k}|} = \frac{2.5}{3} \times (2\hat{i} - 2\hat{j} + \hat{k})$$

$$= \frac{5}{6} \times (2\hat{i} - 2\hat{j} + \hat{k})$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$= \frac{5}{6} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 4 & 1 & -2 \end{vmatrix} = \frac{5}{6} [3\hat{i} + 8\hat{j} + 10\hat{k}]$$



12. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of P and Q respectively, and $\vec{r}$ the position vector of any point R on the line joining P and Q.

$$\vec{r}_1 + \vec{PR} = \vec{r} \text{ or,}$$

$$\vec{PR} = \vec{r} - \vec{r}_1$$

$$\vec{r}_1 + \vec{PQ} = \vec{r}_2$$

$$\text{or, } \vec{PQ} = \vec{r}_2 - \vec{r}_1$$

$$\text{But } \vec{PR} = t\vec{PQ}$$

where t is a scalar

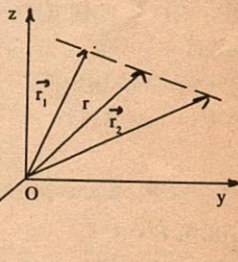
$$\text{Then } \vec{r} - \vec{r}_1 = t(\vec{r}_2 - \vec{r}_1)$$

is the required vector equation of the straight line.

In rectangular coordinates, we have, since

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = t[(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})]$$



$$- (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})] \\ \text{or, } (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = t[(x_2 - x_1)\hat{i} \\ + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

Since, $\hat{i}, \hat{j}, \hat{k}$ are noncoplanar vectors, we have

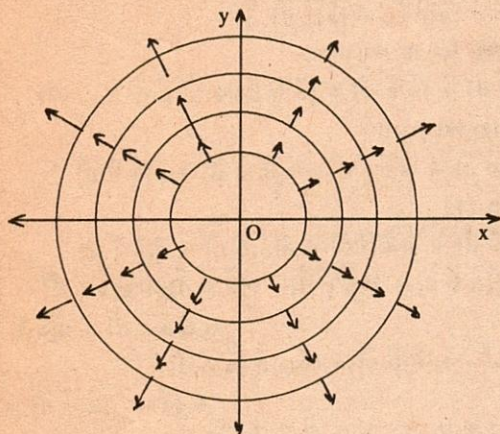
$$x - x_1 = t(x_2 - x_1), y - y_1 = t(y_2 - y_1)$$

$$z - z_1 = t(z_2 - z_1)$$

as the parametric equations of the line, t being the parameter. Eliminating t , the equation become

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

13. At each point (x, y) except $(0, 0)$ of the xy plane there is defined a unique vector $x\hat{i} + y\hat{j}$ of magnitude $\sqrt{x^2 + y^2}$ having direction passing through the origin and outward from it. To simplify graphing procedures, note that all vectors associated with points on the circle $x^2 + y^2 = a^2, a > 0$ have magnitude a . The field therefore appears as shown where an appropriate scale is used.



$$14. \vec{A} = 2\hat{i} + \hat{k}, \vec{B} = \hat{i} + \hat{j} + \hat{k}, \vec{C} = 4\hat{i} - 2\hat{j} + 7\hat{k} \\ \vec{R} \times \vec{B} = \vec{C} \times \vec{B}$$

Taking cross product with $\vec{A}$,

$$\vec{A} \times (\vec{R} \times \vec{B}) = \vec{A} \times (\vec{C} \times \vec{B})$$

$$\text{or, } (\vec{A} \cdot \vec{B})\vec{R} - (\vec{A} \cdot \vec{R})\vec{B} = (\vec{A} \cdot \vec{B})\vec{C} - (\vec{A} \cdot \vec{C})\vec{B}$$

$$\text{or, } (\vec{A} \cdot \vec{B})\vec{R} - (\vec{O})\vec{B} = (\vec{A} \cdot \vec{B})\vec{C} - (\vec{A} \cdot \vec{C})\vec{B}$$

$$\text{or, } \vec{R} = \vec{C} - \frac{(\vec{A} \cdot \vec{C})}{(\vec{A} \cdot \vec{B})} \cdot \vec{B}$$

$$= 4\hat{i} - 2\hat{j} + 7\hat{k} = \frac{15}{3}(\hat{i} + \hat{j} + \hat{k})$$

$$= -\hat{i} + 2\hat{k} - 7\hat{j}$$

$$\text{Alternative : } \vec{R} \times \vec{B} = \vec{C} \times \vec{B}$$

$$\text{Let } \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{or, } (y - z)\hat{i} + (z - x)\hat{j} + \hat{k}(x - y) = -9\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\therefore y - z = -9$$

$$z - x = 3$$

$$x - y = 6$$

$$\vec{R} \cdot \vec{A} = 0$$

$$\therefore 2x + z = 0$$

$$\frac{x - z = -3}{x = -1} \quad (\text{obtained earlier})$$

$$\text{Solving, } y = -7$$

$$z = 2$$

$$\therefore \vec{R} \text{ is } -\hat{i} - 7\hat{j} + 2\hat{k}.$$

15. Let the perpendiculars be concurrent at L .

Let O be the origin of reference

$$\text{Let } \vec{OL} = \vec{l}$$

$$\vec{OA} = \vec{a}, \vec{OQ} = \vec{q}, \vec{OR} = \vec{r}, \vec{OB} = \vec{c}, \vec{OB} = \vec{b}$$

$$\text{Then, } (\vec{l} - \vec{a}) \cdot (\vec{r} - \vec{q}) = 0$$

$$(\vec{l} - \vec{b}) \cdot (\vec{p} - \vec{r}) = 0$$

$$(\vec{l} - \vec{c}) \cdot (\vec{q} - \vec{p}) = 0$$

$$\text{Adding, } \vec{a}(\vec{q} - \vec{r}) + \vec{b}(\vec{r} - \vec{p}) + \vec{c}(\vec{p} - \vec{q}) = 0$$

$$\Rightarrow \vec{p}(\vec{c} - \vec{b}) + \vec{q}(\vec{a} - \vec{c}) + \vec{r}(\vec{b} - \vec{a}) = 0 \quad \dots (1)$$

Let the perpendicular from P to BC , Q to CA meet at $V(\vec{v})$

$$\therefore (\vec{p} - \vec{v}) \cdot (\vec{b} - \vec{c}) = 0$$

$$(\vec{q} - \vec{b}) \cdot (\vec{c} - \vec{a}) = 0$$

$$\text{Adding, } \vec{p}(\vec{b} - \vec{c}) + \vec{q}(\vec{c} - \vec{a}) + \vec{v}(\vec{a} - \vec{b}) = 0$$

$$\Rightarrow -\vec{r}(\vec{a} - \vec{b}) + \vec{v}(\vec{a} - \vec{b}) = 0$$

$$\Rightarrow (\vec{r} - \vec{v})(\vec{a} - \vec{b}) = 0$$

i.e. $\vec{RV}$ is perpendicular to $\vec{AB}$.

Hence, perpendicular from R to $\vec{AB}$ passes through $\vec{V}$. Hence the result.

16. Let $\vec{r}_1, \vec{r}_2, \vec{r}_3$ be the position vectors of P_1, P_2 and P_3 with respect to O' and let $\vec{v}$ be the position vector of O' with respect to O . We seek conditions under which the equation $a_1\vec{r}_1' + a_2\vec{r}_2' + a_3\vec{r}_3' = 0$ will hold in the new reference system.

From figure, it is clear that

$$\vec{r}_1 = \vec{v} + \vec{r}_1'$$

$$\vec{r}_2 = \vec{v} + \vec{r}_2' \text{ so that}$$

$$a_1 \vec{r}_1 + a_2 \vec{r}_2 + a_3 \vec{r}_3 = 0$$

becomes

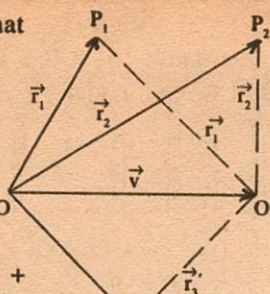
$$a_1(\vec{v} + \vec{r}_1') + a_2(\vec{v} + \vec{r}_2') + a_3(\vec{v} + \vec{r}_3') = 0$$

$$\text{or, } (a_1 + a_2 + a_3)\vec{v} + a_1 \vec{r}_1' + a_2 \vec{r}_2' + a_3 \vec{r}_3' = 0$$

$$\text{The result } a_1 \vec{r}_1' + a_2 \vec{r}_2' + a_3 \vec{r}_3' = 0$$

$$\text{will, hold if } (a_1 + a_2 + a_3)\vec{v} = 0$$

i.e. $a_1 + a_2 + a_3 = 0$ The result can be generalised.



17. The area of a triangle determined by $\vec{R}$ and $\vec{S}$ is $\frac{1}{2} |\vec{R} \times \vec{S}|$

The vectors associated with each face of the tetrahedron are

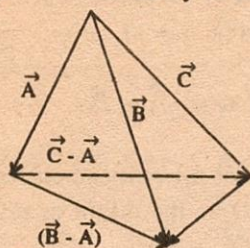
$$V_1 = \frac{1}{2} (\vec{A} \times \vec{B})$$

$$V_2 = \frac{1}{2} (\vec{B} \times \vec{C})$$

$$V_3 = \frac{1}{2} (\vec{C} \times \vec{A})$$

$$V_4 = \frac{1}{2} ((\vec{C} - \vec{A}) \times (\vec{B} - \vec{A}))$$

$$\begin{aligned} \text{Then } V_1 + V_2 + V_3 + V_4 &= \frac{1}{2} [\vec{A} \times \vec{B} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} + (\vec{C} - \vec{A}) \times (\vec{B} - \vec{A})] \\ &= \frac{1}{2} [\vec{A} \times \vec{B} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} + \vec{C} \times \vec{B} - \vec{C} \times \vec{A} - \vec{A} \times \vec{B} + \vec{A} \times \vec{A}] = 0 \end{aligned}$$



The result can be generalised for any closed polyhedra and in the limiting case to any closed surface.

18. Since no three of the vectors are coplanar, $\vec{d} = \lambda \vec{a} + \mu \vec{b} + \gamma \vec{c}$ where λ, μ, γ are nonzero scalars
 $\vec{d} \cdot (\vec{b} \times \vec{c}) = \lambda \vec{a} \cdot (\vec{b} \times \vec{c}) + 0 + 0$

$$\therefore \lambda = \frac{[\vec{d} \vec{b} \vec{c}]}{[\vec{a} \vec{b} \vec{c}]}$$

$$\text{Similarly, } \mu = \frac{[\vec{d} \vec{c} \vec{a}]}{[\vec{a} \vec{b} \vec{c}]}$$

$$\gamma = \frac{[\vec{d} \vec{a} \vec{b}]}{[\vec{a} \vec{b} \vec{c}]}$$

$$\therefore \vec{d} = \frac{[\vec{d} \vec{b} \vec{c}]\vec{a}}{[\vec{a} \vec{b} \vec{c}]} + \frac{[\vec{d} \vec{c} \vec{a}]\vec{b}}{[\vec{a} \vec{b} \vec{c}]} + \frac{[\vec{d} \vec{a} \vec{b}]\vec{c}}{[\vec{a} \vec{b} \vec{c}]}$$

$$\text{or, } [\vec{a} \vec{b} \vec{c}]\vec{d} = [\vec{b} \vec{c} \vec{d}]\vec{a} - [\vec{a} \vec{c} \vec{d}]\vec{b} + [\vec{a} \vec{b} \vec{d}]\vec{c}$$

$$\text{or, } [\vec{a} \vec{b} \vec{c}]\vec{d} + [\vec{a} \vec{c} \vec{d}]\vec{b} = [\vec{b} \vec{c} \vec{d}]\vec{a} + [\vec{a} \vec{b} \vec{d}]\vec{c}$$

To show $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ represent the position vectors of a plane quadrilateral, we assume the result and hence prove the condition

$$\frac{[\vec{b} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{d}]}{[\vec{a} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{c}]} = 1$$

Now, from figure

$$[\vec{AB}, \vec{AC}, \vec{AD}] = 0$$

$$[\vec{b} - \vec{a}, \vec{c} - \vec{a}, \vec{d} - \vec{a}] = 0$$

$$\text{or, } (\vec{b} - \vec{a}) \cdot [(\vec{c} - \vec{a}) \times (\vec{d} - \vec{a})] = 0$$

$$\text{or, } (\vec{b} - \vec{a}) \cdot$$

$$[\vec{c} \times \vec{d} - \vec{c} \times \vec{a} - \vec{a} \times \vec{d}] = 0$$

$$\text{or, } [\vec{b} \vec{c} \vec{d}] - [\vec{b} \vec{c} \vec{a}] - [\vec{b} \vec{a} \vec{d}]$$

$$- [\vec{a} \vec{c} \vec{d}] - 0 - 0 = 0$$

$$\text{or, } [\vec{b} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{d}] = [\vec{b} \vec{c} \vec{a}] + [\vec{a} \vec{c} \vec{d}]$$

$$\text{or, } [\vec{b} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{d}] = [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{c} \vec{d}]$$

$$\text{or, } \frac{[\vec{b} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{d}]}{[\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{c} \vec{d}]} = 1$$

Again, let us assume

$$[\vec{b} \vec{c} \vec{d}] + [\vec{a} \vec{b} \vec{d}] = [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{c} \vec{d}] \quad \dots (1)$$

Using 1st part,

$$\vec{d}[\vec{a} \vec{b} \vec{c}] + \vec{b}[\vec{a} \vec{c} \vec{d}] = \vec{a}[\vec{b} \vec{c} \vec{d}] + \vec{c}[\vec{a} \vec{b} \vec{d}]$$

From (1)

$$\vec{a}[\vec{b} \vec{c} \vec{d}] + \vec{a}[\vec{a} \vec{b} \vec{d}] = \vec{a}[\vec{a} \vec{b} \vec{c}] + \vec{a}[\vec{a} \vec{c} \vec{d}]$$

$$\text{or, } \vec{d}[\vec{a} \vec{b} \vec{c}] + \vec{b}[\vec{a} \vec{c} \vec{d}] - \vec{c}[\vec{a} \vec{b} \vec{d}] + \vec{a}[\vec{a} \vec{b} \vec{d}]$$

$$= \vec{a}[\vec{a} \vec{c} \vec{d}] + \vec{a}[\vec{a} \vec{b} \vec{c}]$$

$$\text{or, } (\vec{d} - \vec{a})[\vec{a} \vec{b} \vec{c}] + (\vec{b} - \vec{a})[\vec{a} \vec{c} \vec{d}]$$

$$+ (\vec{a} - \vec{c})[\vec{a} \vec{b} \vec{d}] = 0$$

$$\text{or, } [\vec{a} \vec{b} \vec{c}](\vec{a} - \vec{d}) + [\vec{a} \vec{c} \vec{d}](\vec{a} - \vec{b})$$

$$+ [\vec{a} \vec{b} \vec{d}](\vec{c} - \vec{a}) = 0$$

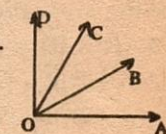
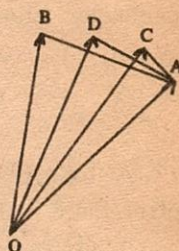
$$\text{or, } \lambda \vec{AD} + \mu \vec{AB} - \gamma \vec{AC} = 0$$

where λ, μ, γ are non zero scalars.

So, $\vec{AD}, \vec{AB}, \vec{AC}$ are coplanar,

Hence, $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ represent

the vertices of a plane quadrilateral.



RUSSIAN CONTEST PROBLEMS

- The difference $\sqrt{140\sqrt{2}-571} - \sqrt{40\sqrt{2}+57}$ is an integer. Find it.
- Find all solutions of the equation $\int_0^\alpha \cos(x+\alpha^2)dx = \sin\alpha$, which belong to the interval $[2; 3]$
- In the scalene triangle ABC the altitudes AP and CQ are dropped from the vertices A and C to the sides BC and AB. The area of the triangle ABC is known to be equal to 18, the area of the triangle BPQ to 2, and the length of the segment PQ to $2\sqrt{2}$. Calculate the radius of the circle circumscribed about the triangle ABC.
- Solve the equation $\sin^2 x = \frac{3}{4}$.
- Find the area of the closed figure bounded by the curves $y = 0$, $y = 20 - 6x - 2x^2$.
- Find the negative terms of the sequence $x_n = \frac{A_{n+4}^4}{P_{n+2} \cdot 4P_n} (n=1,2,3,\dots)$, where A_{n+4}^4 is the number of arrangements, and P_{n+2} and P_n are the numbers of permutations.
- Find the extrema of the function $y = \frac{2x-1}{(x-1)^2}$ and indicate the intervals of increase and decrease.
- Calculate the volume of the solid generated by the rotation about the x-axis of the figure bounded by the curves $y = x^3$, $y = 1$, and $x = 3$.
- Solve the equation $\frac{1}{1+\cos^2 x} + \frac{1}{1+\sin^2 x} = \frac{16}{11}$.
- Find the greatest and the least value of the function $f(x) = 2.3^{3x} - 3^{2x} \cdot 4 + 2.3^x$ on the interval $[-1, 1]$.
- Find the area of the figure bounded by the graphs of the functions $y = x^2 - 4$, $y = 4 - x^2$.
- The radius of the sphere inscribed into a cone is R. Find the volume of the cone if the centre of the sphere circumscribed about the cone coincides with the centre of the inscribed sphere.
- Squares are constructed on the bases AB and CD of the trapezoid ABCD (in the exterior of it). Prove that the straight line connecting the centres of the squares passes through the point of intersection of the diagonals of the trapezoid.
- Solve the inequality $\log x^2(3-2x) > 1$.
- Solve the equation $2\cos x = \sqrt{2 + \sin 2x}$
- Having simplified the equation $\left(\frac{(x^2-4)^{-2} + (x+2)^{-2} - (x-2)^{-2}}{1-8x} \right)^{-1} - (4\sqrt[3]{4})^{\log_4 x^3} = 16$, find its limit as $x \rightarrow 1/8$
- Solve the system of equations $\begin{cases} (3y^2 + 1)\log_3 x = 1 \\ x^{2y^2 + 10} = 27 \end{cases}$
- The sum of the infinitely decreasing geometric progression is equal to the greatest value of the function $f(x) = 3x^3 - x - 76$ on the interval $[0; 3]$; the first term of the progression is equal to the square of its common ratio. Find the common ratio of the

progression.

19. Simplify the expression

$$\left(\left(1 - \frac{1+ab}{1+\sqrt[3]{ab}} \right) \left(\sqrt[3]{ab}(1-\sqrt[3]{ab}) - \frac{(1-ab)(\sqrt[3]{ab}-1)}{1+\sqrt[3]{ab}} \right) \right)^3 - ab$$

20. Solve the equation

$$\cos \frac{\pi-2x}{2} + \sin \frac{5\pi+2x}{2} = \tan \frac{13\pi}{4}$$

21. Solve the inequality $2\sqrt{x^2} \geq (x-1)^2 + 2$.

22. Two sportsmen start simultaneously from diametrically opposite points A and B and run towards each other at constant speeds along a track of circular shape. They first met t seconds later, a metres from B, and for the second time they met at a distance of $2a$ metres from A ($a > 0$). Find the speeds of the sportsmen if the distance means the shortest way along the track.

23. The circles O_1 and O_2 have external tangency at a point A, the segment AB being the diameter of O_1 . The lengths of the segments intercepted by the circles on some straight line passing through the point B are equal to 2 cm, 3 cm, and 4 cm, reckoning from the point B. Find the radii of the circles.

24. A cylinder is inscribed into a regular quadrangular pyramid whose lateral faces are at the angles ϕ to the plane of the base (one base of the cylinder lies in the plane of the base of the pyramid and the circumference of its second base has one point in common with every lateral face of the pyramid). The radius of the base of the cylinder and its altitude are equal to r . Calculate the volume of the pyramid. At what angle ϕ is the volume of the pyramid the least?

25. Solve the equation

$$\cos^2 \left(\frac{\pi}{3} \cos x - \frac{8\pi}{3} \right) = 1.$$

SOLUTIONS

1. -10

2. $\left\{ \sqrt{2\pi}, \frac{\sqrt{8\pi+1}-1}{2} \right\}$

3. $9/2$

4. $\pm \frac{\pi}{3} + \pi k, k \in \mathbb{Z}$

4. $114\frac{1}{3}$

6. $x_1 = -63/4, x_2 = -23/8$

7. The point $x=0$ is point of minimum, the intervals of decrease are $(-\infty, 0)$ and $(1, \infty)$. The interval of the increase is $(0, 1)$.

8. $310\frac{2}{7}\pi$

9. $\pm \frac{\pi}{12} + \frac{\pi}{2}k, k \in \mathbb{Z}$

10. 24; 0

11. $64/3$

12. $3\pi R^3$

13. Hint: Under the homothetic transformation with centre $[AC] \cap [BD]$ and the coefficient $-[CD]/[AB]$, the square constructed on $[AB]$ passes into a square constructed on $[CD]$

14. $(-3; -1)$

15. $\frac{1}{2}\arctan 2 + 2\pi k, \frac{1}{2}\arctan 2 - \frac{\pi}{2} + 2\pi l (k, l \in \mathbb{Z})$

16. $-1/8$

17. $\{(\sqrt[3]{3}; 1), (\sqrt[3]{3}; -1)\}$

18. $\sqrt{3} - 1$

19. 0

20. $2\pi k, \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$

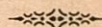
21. $(1; 3)$

22. $\frac{4a}{t} \text{ m/s}, \frac{a}{t} \text{ m/s}$

23. $\frac{\sqrt{30}}{5} \text{ cm}, \frac{\sqrt[3]{30}}{10} \text{ cm}$

24. $V = \frac{4}{3}\pi r^3 \frac{(\tan \phi + 1)^3}{\tan^2 \phi}, \phi = \arctan 2$

25. $\pi + 2\pi k, k \in \mathbb{Z}$



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HISTORY OF MATHEMATICS

The Invention of Algebra

— Prof. R.P. Sinha

Contd. from previous issue

Nowadays we simply say that the length of AC in Figure 4 is “the square root of 2 inches,” and we write this as “ $\sqrt{2}$ inches.” The symbol $\sqrt{}$ is of comparatively recent origin. To understand its origin we must first see how the word “root” came into mathematics. The Greeks did not use the expression. They spoke of the “side” of a “square number.” Thus they would call 3 the “side” of the square number 9; 4 the “side” of the square number 16, and so on. As we have seen, the Arab obtained much of their mathematics from the Greeks. When however, they adopted the simple and convenient Hindu number symbols, their own mathematical reasoning became based to a greater extent on “number” than on geometrical figures. Thus it came about that instead of speaking of the “side” of 16 being 4, they dropped the geometrical notion of a square and conceived of a number as “growing” out of the “root” 4.

After a book by al-Khowarizmi on number-reckoning had become known in Europe, European mathematicians took over the Arab idea of a “root” and translated the Arabic word by the Latin *radix* (compare “radish”; “radical,” one who, in his own opinion at any rate, goes to the root of every problem). They also began to speak of *extracting* or “dragging out” the root rather than of finding it.

Towards the end of the Middle Ages, during the stage in algebra’s development when abbreviations were being introduced, this word *radix* was abbreviation for “recipe” in a medical prescription.

Thus R25 was 5. About a century before the introduction of printing we find a small r being written in place of this R, thus r25 = 5, and it is through that our modern “radical sign” $\sqrt{}$ is simply a copy of the small letter r as written by some copyist before printing was introduced. That is why we now write “the square root of 2” as $\sqrt{2}$.

To return to the later Pythagoreans. They found that there were countless other numbers of the same type as $\sqrt{2}$ which could not be expressed exactly in numbers. This applied to the “side” of any number that was not a “square number.”

There is an interesting story connected with the name given by Greek mathematicians to numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and so on. Greek mathematicians used the word *logos*, which meant “a word” and also “the mind behind a word” for any number that could be expressed as a ratio, such a number being one that could be expressed as a ratio between two numbers, was called *a-logos*, that is, “not logos,” “the mind behind a word” for any number that could be expressed as a *ratio*, such a number being one that their minds could grasp. In fact, the Greek word for “ratio” was *logos*. Any number like $\sqrt{2}$, which, as we have seen, cannot be expressed as a ratio between two numbers; was called *a-logos*, that is, “not logos,” “not a ratio number.”

We must now jump ahead about 1000 years in order to follow up the story of *a-logos*. The Arab mathematician al-Kho-warizimi made use of an Arabic translation of the work of the some Greek

mathematician, who naturally used the word *a logos*, in its primary instead of its technical sense, as indicating "not ratio-nal." Whoever translated the work into Arabic took the word *a logos*, in its primary instead of its technical sense, as meaning "without a word" and translated it by the Arabic word meaning "deaf." So al-Khowarizmi came to call numbers like $\sqrt{2}$, $\sqrt{3}$, etc. by this Arabic word. Some three hundred years later the European translator Gherado rendered into Latin a book written by al-Khowarizmi. He found certain numbers described in the Arabic as "deaf", so he translated this by the Latin *surdus* "deaf." To this day we still call a number like $\sqrt{2}$ a "surd, with its meaningless reference to deafness.

The story of a-logos numbers has taken us for a moment into the second stage of the story of algebra. Let us retrace our steps to the time of the Golden Age of Greek mathematics.

Greek mathematicians gave their Science of Number the Greek word *arithmetike*, since *arithmos* meant "number," *techne*, "science." We must not to be misled by our present use of the word "*arithmetic*" into supposing that this arithmetic had any connection originally with the simple number-reckoning was known as *logistic* by the Greeks and was considered by mathematicians as unworthy of their study and attention being connected with the every day calculations that were made on an abacus. Eventually, the title *arithmetike* or *arithmetic*, as it came to be written in the Middle Ages, ceased to be used for the "Science of Numbers," being replaced, as we shall see, by the Arabic word *algebra*. The title "*arithmetic*" did not reappear until the eighteenth century, when, strange to say, it stood for its old "superior" branch of mathematics, for it was then used for the once-despised *logistic*, number of reckoning. Since the eighteenth century the word

"arithmetic" has been used thus, in the sense that it is used today. It is possible though not certain, that this surprising change in the meaning of a word came about through a mistake regarding its derivation. The word is sometimes found in the eighteenth century as *ars-metrica*, as though it came from the Latin *ars*, "art," *metrica*, "measuring," instead of from the Greek *arithmos*, *techne*, as we saw. Whatever may be the correct explanation of this surprising change in meaning, the word is now always connected with the elementary and practical *art of measure*.

To return to the Greek mathematicians and their work. During the five centuries that passed after Euclid wrote the *Elements*, they began to work certain problems by using numbers rather than geometric figures. Most of these problems were the kind we should now say lead to equations. It must be remembered, however, that those who worked them had no algebraic symbolism such as we have today, but wrote out their solutions in words written in full.

There is famous collection of ancient Greek epigrams, or short poems that usually end in a witty phrase, called the *Greek Anthology*. Among these epigrams are forty-six "number problems," some of them thought to have been written at least a thousand years before they were collected in the *Anthology* about A.D. 500. All these problems lead to one kind of equation or another. Here is one of them:

"One third or a number of apples is to be given to one man, one-eighth to a second man, one fourth to a third man, one-fifth to a fourth man; a fifth man is to get 10 of them. How many apples will be required?"

If we let the symbol *x* stand for the unknown quantity, that is, for the number of apples required, and use our modern algebraic "shorthand" we can

at once translate this problem into the equation $\frac{x}{3} + \frac{x}{8} + \frac{x}{4} + \frac{x}{5} + 10 = x$. This is very easily solved if we multiply every term in the equation by 120, the smallest number that contains 3, 8, 4 and 5 exactly, or the "lowest common multiple" of these numbers.

But a problem like this would be quite difficult if we did not have our modern symbols and methods. Right up to the seventeenth century, problems like this were worked by methods very similar to those set out in the Ahmes papyrus.

Let us glance back at the Egyptian problem we mentioned in the beginning. It is even easier than the one just quoted from the *Greek Anthology*, but all the same, there is no easy way of solving it unless we use our modern symbols and methods.

Here in a brief outline is the ancient method by which the Egyptian problem would have been solved right up to the seventeenth century. Bear in mind that our modern number symbols and methods of computation make it much easier than it would have been before their introduction.

The Egyptian problem was to find a number such that if that number is added to one-seventh of itself, the result will be 19.

First, choose any number that seems to be likely to fit some of the given facts. In the Ahmes papyrus, the number 7 is chosen. Now see whether this number fits in with all the given facts. Clearly, 7 will not do for the unknown quantity, since if you add 7 to one-seventh of itself you get only 8, and not 19.

But 19 is $2\frac{3}{8}$ times as great as 8, so the true value of the unknown will be $2\frac{3}{8}$ times as great as the "false" answer 7 with which we started the calculation. Now, $2\frac{3}{8}$ times 7 gives $16\frac{5}{8}$, the true value of the unknown quantity.

This method was known during the Middle Ages as the *Rule of False Position* or simply the *rule of false*. This method of solving an equation, by which a start is made by supposing a number (you know is most probably false) to be the true answer, was used until some three hundred years ago. Try working the "apple problem" we mentioned just now by this method, and see how long it takes.

We now pass on to the second stage in the story of algebra's growth. Round about the year A.D. 250, an Alexandrian mathematician named Diophantus was flourishing. We do not know when he was born, and we have scarcely any knowledge about his personal life except what is found in an epigram in the *Greek Anthology*. This says that his boyhood lasted one-sixth of his life; his father's age, and the father died 4 years after his son. How old was Diophantus when he died?

Since it would be interesting to know at least how long this mathematician lived, we better work this problem, though we shall not use the *Rule of False* employed by all those who calculated his age during the Middle ages!

Let the symbol x represent the number of years Diophantus lived, and let the sign $\therefore$ stand for "there fore"

$$\therefore \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x.$$

Now multiply each term by 84, the lowest common multiple of all the denominators.

$$\therefore 14x + 7x + 12x + 420 + 42x + 336 = 84x$$

$$\therefore 756 = 9x$$

$$\therefore x = 84$$

So, if the epigram is based on fact, we do at least know that Diophantus lived to a ripe old age.

During those eighty-four years he won himself the title, given long after his death of "the father of

algebra." It is difficult to see how this claim can be substantiated. It is true that he wrote a book, *the Arithmetica*, which is unquestionably one of the world's greatest mathematical books. Some fourteen centuries later, it was to help mould the mind of Fermat, one of the greatest of French mathematicians. Originally, it consisted of thirteen parts, like Euclid's *Elements*, but only six of them have survived. It deals with number-problems which lead to equations, some of them very difficult one's, involving what we now call quadratic, cubic and bi-quadratic equations, or equations involving the second, third, or fourth power of the unknown. Many of these equations are "indeterminate," that is, they each consist of an equation which contains more than one unknown. So thorough was Diophantus in dealing with these indeterminate equations that they are still sometimes called "Diophantine." But all the same, as has been pointed out by two of the greatest authorities on Greek mathematics, Dr. Nesselmann and Sir Thomas Heath, Diophantus was by no means the first mathematician to work equations such as these. Nor can it be said that the methods he employed were responsible for modern algebraic technique; his was not developed until fifteen hundred years after this time.

The great, outstanding characteristic of modern algebra is that it provides mathematicians with a simple, concise language, in shorthand form, in which to express and clarify mathematical concepts and processes. This mathematical shorthand makes use of symbols and signs which have no outward connection, so far as their appearance goes, with the words of things they represent. It can be mastered with far less strain on the memory than can the shorthand used by a stenographer, yet it enables those who understand it to state and solve problems

without using any ordinary words, except for an occasional connecting word such as a system, though this claim is often made on his behalf. The so-called symbols used by him were merely abbreviations, which entirely lack the unlimited "generality" and power of our modern symbols.

As we have seen, there are three distinct states in the story of the evolution of algebra. The first stage extended from the time of the ancient Egyptians to that of Diophantus, around A.D.250. In this stage, the solution of a problem was written out in the word and sentences in full, like a piece of prose, or a philosophical argument. During the last four centuries of this first stage in the development of arithmetic, Greek mathematicians come to use the word *arithmos* (Greek word for "number") for the unknown quantity in a problem, writing the word in full, where we should use a symbol like x .

Diophantus apparently grew weary of repeatedly writing the word *arithmos* for an unknown quantity; he abbreviated it to its first two letters, a, r , writing them, of course, in Greek, as $\alpha\rho$. Having to write this abbreviation thousands of times, in course of time he merged the letters into φ , and then into ς , and finally into ζ . [This is the theory put forward by Sir Thomas Heath. It is not accepted by all historians of mathematics.]

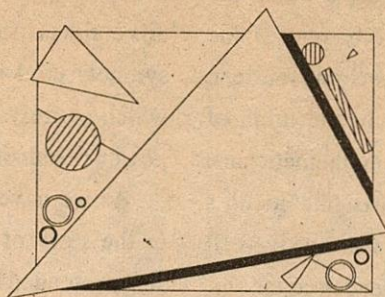
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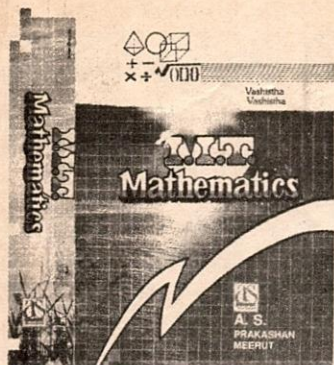
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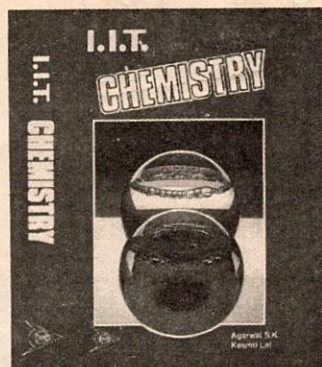
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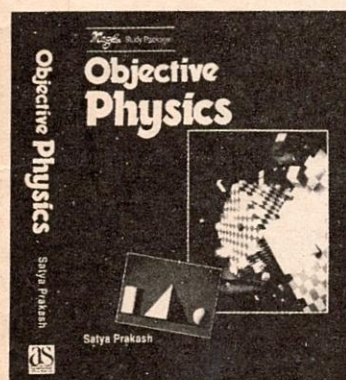
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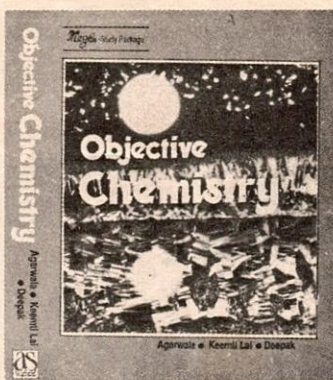
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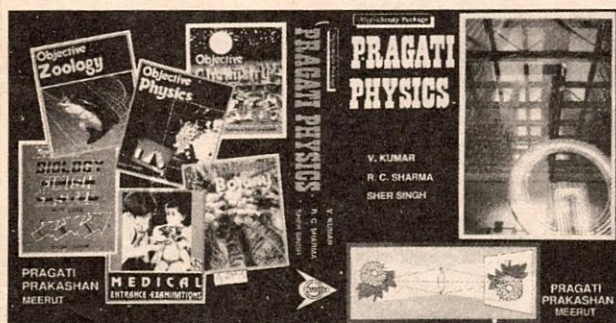
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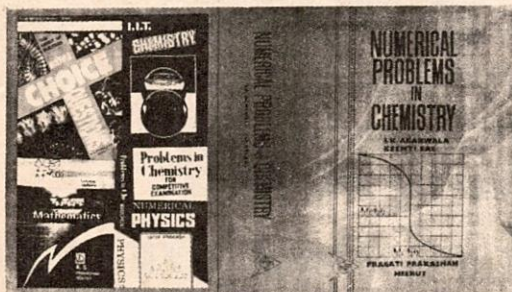
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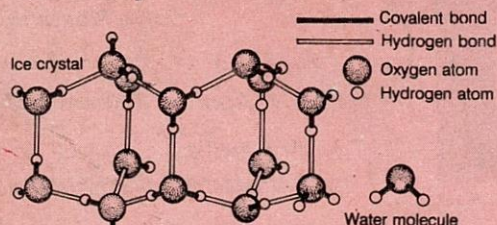
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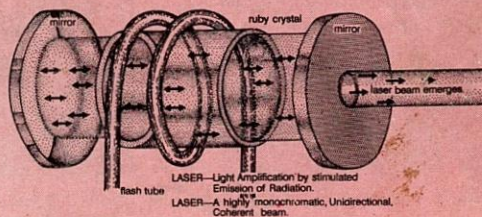
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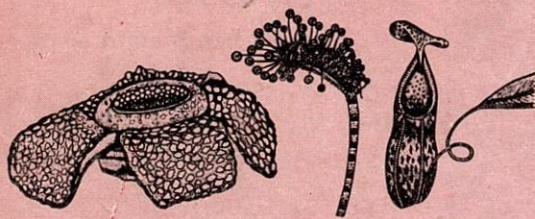


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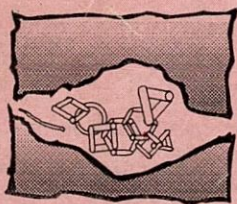
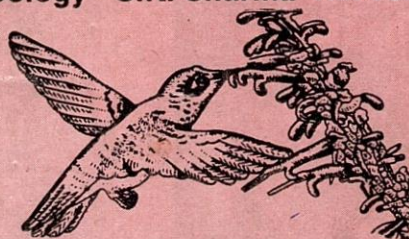
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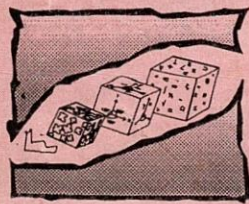
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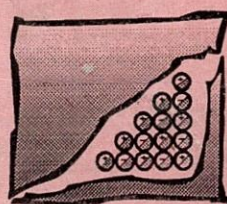
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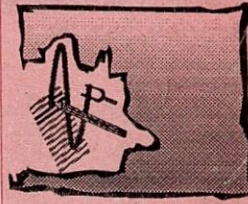
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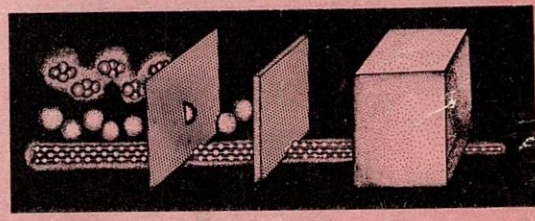
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